Problem #1: Kepler's problem

The solution to Kepler's problem was a great achievement of the human kind. Before that people thought the celestial motion is propelled by invisible angels flapping wings. Now the motion in the sky is no more special than the motion on the ground.

A general planet orbit around the sun is an ellipse. Suppose the gravity $\mathbf{F} = -\frac{\gamma}{r^2}\hat{e}_r$. Please prove the following results.

1. Express the half-major axis a, half-minor axis b, the half chord length c passing the focus and perpendicular to the major axis, the eccentricity e in terms of the physical observable of the total energy E and the magnitude of angular momentum l.
2. Check a and c are special which solely depend on one quantity, respectively.
3. For a given planet, if its energy E is fixed, which orbit has the maximal orbital angular momentum l? If its orbital angular momentum l is fixed, which orbit has the smallest energy E? (You need to prove your statements).
4. Prove Kepler's 3rd law for the general elliptic orbit.

These are important results, and I hope you remember them.

Problem #2: Cosmic velocities

Read the lecture notes on cosmic velocities which I do not have time to teach. The 3rd cosmic velocity is significant, which is the minimal velocity to escape the solar system. The earth is the cradle of the human kind, and in this sense the human kind is still in the childhood. Only when the human kind can escape the solar gravity, we can be recognized as an adult civilization and participate interstellar affairs.

In fact, in the science fiction *"Three-body problem" Vol III*, it made a mistake. It said the solar system is facing the danger of interstellar strikes from high-level extraterrestrial civilizations based on the *"Dark Forest"* principle. Nevertheless, there is a remedy if the solar system could make a safety announcement: If the light velocity could be lowered to the critical value $v_c = 16.7km/s$, i.e., the 3rd cosmic velocity, the solar system would become a black hole. Nothing can go out, and the solar system will become isolated from other stars. Although the life would be boring, the solar system would have no threat to any extraterrestrial civilization such that strikes would not be launched.

1) Work out the first cosmic velocity.

2) Work out the 2nd cosmic velocity and show its relation with the 1st one.

3) Work out the 3rd cosmic velocity.

Here is a simplified approach. In the earth frame, the aircraft needs to first escape the earth gravity field, and beyond that it still possesses a sufficient amount of velocity to escape the gravity field of the sun. You need to figure out a way to transform back and forth between the earth and the sun frames to find the answer.

4) The lecture note tells you how to solve the 3rd cosmic velocity from the reference frame of Sun. It is more complicated, but tells you more clearly on how energy is conserved. Please do this problem again using the method of in the lecture note.

5) After you have done the above steps, you will understand that Liu actually misunderstood the meaning of the 3rd cosmic velocity. How? (We still assume that light velocity should be independent of reference frame, which is called the invariance of light velocity based on the relativity principle.)

Problem #3: Shallow impact - Double Asteroid Redirection Test

Recently, NASA launched the DART project. This is a historical event that the human kind changed the orbit of a celestial object. It is quite similar to the plot in *"Deep Impact""*, a science fiction movie in 1990's, although the impact scale is much smaller. Nevertheless, this is a beginning step of a long-march.

NASA picked up a binary asteroid system in which the small asteroid orbits around the main one. A spacecraft was launched to collide the small asteroid to change its orbit. Now imagine that you are the chief physicist in charge of this project. You would need to estimate the consequence of the impact and compare with the observation.

1) The primary astroid is called *Didymos* with the mass 528 billion kg, and the small one is called *Dimorphos* with the mass 4.8 billion kg. They interact with each other via gravity. The *Dimorphos*' periodicity around *Didymos* was measured as 11 hours 55 minutes before the collision, which can be observed through telescope since *Dimorphos* will block *Didymos* once in one period.

Please calculate the orbital radius R by assuming a circular orbit, and the orbiting speed v. You will see how slow of the orbiting speed is which may at the same order of a tortoise.

2) The spacecraft's mass is about 600kg, and before the impact its speed is 22,530km/hour. Assume that the impact is completely inelastic that spacecraft sticks to Dimorphos after the impact. How to design the impact such that the orbital period of Dimorphos can be shortened most significantly? Before the impact, NASA estimated the period would be shortened by 73 second. Compare your results with NASA's estimation, which should have a big difference. Can you think about possible reasons to the difference (There could be many reasons, just think some.)?

Actually, NASA observed that the period after the impact is shortened by 32 minutes, which is 25 times larger than the estimation. One possibility is the impact was like an explosion. Rocks were kicked out to the space which carry further momentum and further slow down Dimorphos.

Problem #4: The appearance of π

Consider the collisions between two blocks with masses M and m, respectively, which are put on a slippery floor in front of a wall shown in Fig. 1. $M > m$ is assumed. The big block M initially moves toward the wall with v_0, and the small block m is initially at rest. After the first collision, m moves toward and collide with the wall and then bounced back to collide with M again. The above process repeats, and then M will receive enough impulse to switch its direction of motion. Finally, m can not catch up with M and there will be no collision. The number of collisions experienced by m are denoted as N (Including those collisions between wall and m). All collisions are assumed to be elastic.

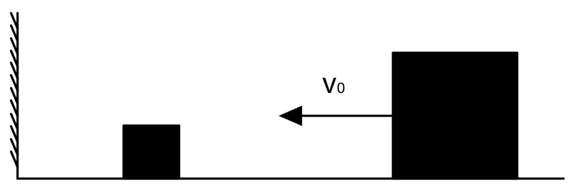

Figure 1: The collision processes between M and m.

1) Obviously N should be determined by the initial settings of the system. Based on dimension analysis, find a rough expression for N.

2) Assume $M = m$. Calculate N by brutal force.

3) Table 1 shows numerical results of N under different mass ratio M/m. You may find some patterns: N always have the same digits with π ! Feynman once said: every time π appears in an expression, there should be a circle somewhere. The circle here arises from the kinetic energy conservation.

M/m	10^2	10^4	10^6	10^8	10^{10}
N	31	314	3141	31415	314159

Table 1: The collision times and the mass ratio.

4) With the help of this circle, give the approximate expression of N. In the limit of $M/m \to \infty$, explain the pattern you find in the above table.

Problem #4: The falling chain problem

A uniform flexible chain with a linear density ρ falls upon a platform from a hanging position. Such a chain is initially suspended by one end with the lower end just contacting the platform. When the chain has fallen a distance s and this length of chain has collapsed upon the platform.

1) During the falling process, does the mechanical energy conserve or not?

2) Please calculate the upward force F that platform exerts on the chain as a function of the fallen distance s.