11

Relativity – without light

So far we have assumed the relativity principle, which was attributed to Galileo, that all inertial frames are the equivalent. Whenever we think about a physical law, it should be checked whether it is the same or not when looked at from a different reference frame. It is the most universal properties of nature that physicists care most. In Galileo and Newton's minds, time is absolute which passes uniformly at an equal pace in all the inertial reference frames. The space-time transformation between two different reference frames, say, the lab frame on the ground and the moving frame of a train, obeys the Galilean transformation. This sound very natural, and for many thousands of years, people took it for granted. Although in legends, it often said that one day in the heaven equals one years in the human world, nevertheless, this was not a scientific statement. Newton's laws of motion are consistent with the Galilean space-time transformation. So far so good.

Historically, the special relativity was developed after the establishment of Maxwell equations. Naturally, such a fundamental achievement should also be examined critically by reference frame transformations. Many prominent physicists, including mathematicians, contributed to the establishment of the special relativity, not just Einstein. Prof. Xiaofeng Jin at Fudan university published a series of article in "*Physics*", a journal of Chinese Physical Society, elucidating the original contributions from Poincaré, the French mathematician and physicist, which was largely overlooked by the physical community. Many revolutionary concepts were actually already proposed by Poinaré before Einstein.

Traditionally, the derivation of the relativistic space-time transformation, i.e., the Lorentz transformation, is based on two postulates. One is the relativity principle, and other is the light velocity invariant. The latter is often criticized that since it is unnatural to juxtapose a concrete velocity of light with the fundamental principle of relativity.

Nevertheless, there have been progresses in achieving relativity without light in literature, which will be summarized in this lecture. We will try not to use knowledge of electromagnetism. Only based on reasonable assumptions such as the homogeneity, smoothness, and isotropy of space and time, we are able to derive that only Lorentz, Galilean, and rotation-like transformations are possible, which corresponds to the hyperbolic, parabolic, and elliptic subgroups of $SL(2, R)$. Nevertheless, if we further impose the requirement of causality, i.e., one cannot travel back to a time before he/she was born, only Lorentz and Galilean transformations are allowed. If we further abandon the perspective of instantaneous interactions, or, the signal prorogation velocity has an upper limit, then Lorentz transformation is the only choice.

11.1 Space-time transformation

For simplicity, consider the case of 1+1 dimensions, i.e., the 1 spatial dimension together 1 temporal dimension. The reference frame F' moves to the right relative to the reference frame F at the velocity u as shown in Fig.11.1. An event is represented by the space-time coordinate (x, t) in the frame F, and (x', t') in the frame F'. When the origin of x in F and the orgin of x' in F' coincide, we set $t = 0$ in F and $t' = 0$ in F'. In other words, $(x', t') = (0, 0)$ if and only if $(x, t) = (0, 0)$.

In general, we assume that the space-time coordinates of F and F' are related by a linear transformation, which is represented in the matrix form

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = R \begin{pmatrix} x \\ t \end{pmatrix}. \tag{11.1}$$

R is a 2×2 matrix represented as,

$$R = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{11.2}$$

The matrix elements A, B, C and D should only depend on the relative velocity u, i.e., every set of (x, t) in F transforms according to the same matrix R to (x', t') in F'.

R should be reversible, which means that for a given set of coordinates (x', t') in F', we should also be able to identify a point (x, y) in F. After expanding Eq. 11.1, we arrive at

$$\begin{aligned} Ax + Bt &= x' \\ Cx + Dt &= t'. \end{aligned} \tag{11.3}$$

It can be solved as

$$x = \frac{1}{\Delta}(Dx' - Bt')$$
$$t = \frac{1}{\Delta}(-Cx' + At'), \tag{11.4}$$

where Δ is the determinant of R defined as $\Delta = \det(R) = AD - BC$. After be cast into the matrix form, the inverse transformation reads

$$\begin{pmatrix} x \\ t \end{pmatrix} = R^{-1} \begin{pmatrix} x' \\ t' \end{pmatrix}, \tag{11.5}$$

where R^{-1} is the inverse matrix of R as

$$R^{-1} = \frac{1}{\Delta} \begin{pmatrix} D & -B \\ -C & A \end{pmatrix}. \tag{11.6}$$

The transformations Eq. 11.1 and Eq. 11.5 not only apply for a single event, but also applies for the difference between two space-time events. Consider two events with coordinates (x_1, t_1) and (x_2, t_2) in F, and their coordinates (x_1', t_1') and (x_2', t_2') in F'. It is obvious to show that

$$\begin{pmatrix} \Delta x' \\ \Delta t' \end{pmatrix} = R \begin{pmatrix} \Delta x \\ \Delta t \end{pmatrix}, \tag{11.7}$$

$$\begin{pmatrix} \Delta x \\ \Delta t \end{pmatrix} = R^{-1} \begin{pmatrix} \Delta x' \\ \Delta t' \end{pmatrix}, \tag{11.8}$$

where $\Delta x = x_2 - x_1$, $\Delta t = t_2 - t_1$. $\Delta x'$ and $\Delta t'$ are defined similarly.

11.2 Constraints to the matrix elements

Now we determine the matrix elements step by step based on common senses.

We first explore the physical meaning of the relative velocity u. Consider a point at rest in F', say, its spacial origin O'. During a time interval $\Delta t'$, it does not move, i.e., $\Delta x' = 0$. But relative to F, it moves at the velocity of u relative to F. According to Eq. (11.8),

$$u = \frac{\Delta x}{\Delta t} = \frac{D\Delta x' - B\Delta t'}{-C\Delta x' + A\Delta t'}|_{\Delta x'=0} = -\frac{B}{A}. \tag{11.9}$$

Similarly, the origin O of the F frame, which is at rest in F, moves at $-u$ relative to F'. According to Eq. (11.7),

$$-u = \frac{\Delta x'}{\Delta t'} = \frac{A\Delta x + B\Delta t}{C\Delta x + D\Delta t}|_{\Delta x=0} = \frac{B}{D}. \tag{11.10}$$

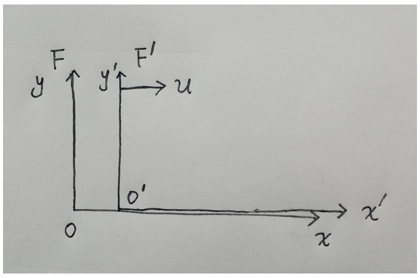

Figure 11.1 Two reference frames F and F'. F' moves at the velocity u relative to F.

Compare above two expressions, we arrive at

$$A = D, \quad B = -Au = -Du. \tag{11.11}$$

Next we examine the composition of two velocities. Consider a set of three frames F_1, F_2, and F_2'. F_2 moves at the velocity u relative to F_1, and F_2' moves at the velocity u' relative to F_1. We ask what are the relative velocity F_2' to F_2, and reversely the one of F_2 to F_2'?

Check the origin O_2' of F_2' moving at u' relative to F_1. Hence, the coordinates of O_2' in F_1 satisfy

$$\frac{x_{O_2',F_1}}{t_{O_2',F_1}} = u'. \tag{11.12}$$

Transform to the frame of F_2, the coordinates of O_2' therein become

$$\frac{x_{O_2',F_2}}{t_{O_2',F_2}} = \frac{A(u)x_{O_2',F_1} + B(u)t_{O_2',F_1}}{C(u)x_{O_2',F_1} + D(u)t_{O_2',F_1}} = \frac{A(u)u' + B(u)}{C(u)u' + D(u)}. \tag{11.13}$$

According to Eq. (11.11), we arrive at

$$v_{F_2' to F_2} = \frac{x_{O_2',F_2}}{t_{O_2',F_2}} = \frac{u' - u}{u'\frac{C(u)}{A(u)} + 1}. \tag{11.14}$$

Similarly, we can derive the velocity of F_2 relative to F_2' just by switching u and u',

$$v_{F_2 to F_2'} = \frac{x_{O_2,F_2'}}{t_{O_2,F_2'}} = \frac{u - u'}{u\frac{C(u')}{A(u')} + 1}. \tag{11.15}$$

Followed by the common sense, $v_{F_2' to F_2} = -v_{F_2' to F_2}$, then $\frac{uC(u')}{A(u')} = \frac{u'C(u)}{A(u)}$, which yields

$$\frac{C(u')}{u'A(u')} = \frac{C(u)}{uA(u)}. \tag{11.16}$$

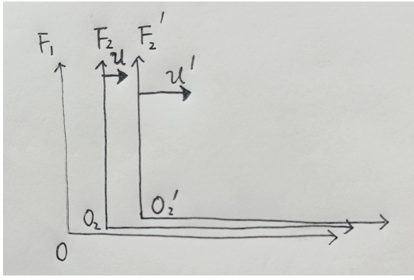

Figure 11.2 Three reference frames F_1, F_2, and F_2'. F_2 and F_2' move at the velocities u and u' relative to F, respectively.

Since the variables are already separated, both sides must equal to a constant K independent on u and u', i.e.,

$$\frac{C(u)}{uA(u)} = K. \tag{11.17}$$

It can be simply checked that K carries the dimension of the square of velocity.

11.3 Special linear group and its subgroups

The space-time transformation matrix R actually forms an algebraic structure, called group. A group is a set of elements satisfying the following conditions:

(i) They are closed under multiplication. In other words, for two transformations R_1 and R_2, their product $R = R_2 R_1$ remains a space-time transformation.
(ii) The existence of an identity element, i.e., $R = I$, which means the two frames are the same.
(iii) Reversibility: For each transformation R, its inverse is defined as R^{-1}.

Furthermore, R has the property of $\det R = 1$ based on the relativity principle proved as below.

Consider two events E_1 and E_2, which take place at the same place in the Frame F with a time difference τ, i.e.,

$$\Delta x = 0, \Delta t = \tau. \tag{11.18}$$

Then in the frame F' which moves at the velocity u relative to F, the time interval transforms as

$$T = \Delta t' = D(u)\tau. \tag{11.19}$$

Similarly, if two events E_3 and E_4 occurs at F' at the same place $\Delta x' = 0$ but the time difference $\Delta t' = \tau$. Then we expect that the corresponding time difference in the F frame should also be T. The reason is that F moves at the velocity $-u$ relative to F'. A relative velocity of $-u$ can be obtained by performing a spacial reflection to the velocity of u, which flips the sign of velocity and displacement but maintain time invariant. Then

$$T = \Delta t = \frac{A(u)}{\Delta}\tau. \tag{11.20}$$

By comparing Eq. (11.19) and Eq. (11.20), since $A = D$, we conclude that

$$\det R = 1. \tag{11.21}$$

All the 2×2 matrices with $\det R = 1$ form the group $SL(2, R)$ – the special linear group. Nevertheless, the space-time transformation matrix $R(u)$ only has one parameter, hence, it is only a one-parameter subgroup of $SL(2, R)$. We need to determine what kind of subgroup it is.

Since $\det R = 1$, we express the transformation matrix as

$$R = \frac{1}{1 + Ku^2}\begin{pmatrix} 1 & -u \\ Ku & 1 \end{pmatrix}. \tag{11.22}$$

Certainly, if $K = 0$, the transformation simply reduces back to the case of Galilean transformation. In this case, the transformation matrix is simply

$$R = \begin{pmatrix} 1 & -u \\ 0 & 1 \end{pmatrix}. \tag{11.23}$$

The transformation matrix has the upper triangular form, and this subgroup of SL(2,R) is called the parabolic type.

Actually, we have more possibilities that K can be finite. In order to make the matrix element dimensionless, we define $K = \pm c^{-2}$, where $\pm$ corresponds to the possibilities of $K > 0$ and $K < 0$, respectively. We combine (x, ct) as the space-time coordinates such that

$$\begin{pmatrix} x' \\ ct' \end{pmatrix} = R\begin{pmatrix} x \\ ct \end{pmatrix}, \tag{11.24}$$

where R is a dimensionless matrix

$$R = \frac{1}{\sqrt{1 \pm (u/c)^2}}\begin{pmatrix} 1 & -u/c \\ \pm u/c & 1 \end{pmatrix}. \tag{11.25}$$

The case of $K > 0$ corresponds to a rotation-like transformation parameterized as

$$\sin\theta = \frac{u/c}{\sqrt{1+(u/c)^2}}, \quad \cos\theta = \frac{1}{\sqrt{1+(u/c)^2}}. \tag{11.26}$$

Then the transformation matrix is represented by

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{11.27}$$

Apparently, this forms a 2D rotational group named as $SO(2)$, i.e., the special orthogonal group. This is the elliptical kind subgroup.

On the contrary, the case of $K < 0$ corresponds to a hyperbolic transformation, which can be parameterized as

$$\sinh\theta = \frac{u/c}{\sqrt{1-(u/c)^2}}, \quad \cosh\theta = \frac{1}{\sqrt{1-(u/c)^2}}. \tag{11.28}$$

The transformation matrix is parameterized as

$$R(\theta) = \begin{pmatrix} \cosh\theta & -\sinh\theta \\ -\sinh\theta & \cosh\theta \end{pmatrix}. \tag{11.29}$$

This is a hyperbolic type subgroup, and Eq. (11.29) is just the Lorentz transformation.

11.4 Consequence of causality

Except the Galileo transformation, we examine the other two possibilities – the elliptical one and the hyperbolic one. We will use the causality to rule out the elliptical transformation, and keep the hyperbolic one as physical.

If the elliptical transformations were the case, we could design a series of reference frames of F_i, $i = 0, 1, 2, 3, ...$, such that F_i moves at the velocity of u relative to F_{i-1}. We parameterize $\tan\theta = u/c$, and then the transformation between F_{n+1} and F_1 is

$$R_{n+1,1}(\theta_n) = R_{n+1,n}(\theta)...R_{2,1}(\theta) = \begin{pmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{pmatrix}. \tag{11.30}$$

Suppose we begin with a small angle of θ, then as n goes large, we would have that $n\theta > \pi/2$, such that $\cos n\theta < 0$.

Now consider two events $E_{1,2}$ in the frame F with $\Delta x = 0$ and $\Delta t = \tau > 0$. E_1 could be the reason and E_2 the consequence, say, E_1 is the birth of a baby, and E_2 is that he/she comes to the same place after growing up. However, in the frame of F_{n+1}, we have

$$\Delta t' = \tau \cos n\theta < 0, \tag{11.31}$$

which means that in this frame the consequence occurs earlier than the reason. This violates the causality, hence, is not allowed!!!

In contrast, the hyperbolic transformations do not have this difficulty as long as $u < c$. In this case,

$$R_{n+1,1}(\theta_n) = R_{n+1,n}(\theta)...R_{2,1}(\theta) = \begin{pmatrix} \cosh n\theta & -\sinh n\theta \\ -\sinh n\theta & \cosh n\theta \end{pmatrix}. \tag{11.32}$$

Since $\cosh n\theta$ is always positive, the causality is not violated as in the previous example.

Naturally, for the hyperbolic case, the velocity c is the upper limit of the physical velocity. Its physical meaning is unclear within the mechanics context itself. We still cannot rule out the possibility of the Galilean transformation.

If we further abandon the perspective instantaneous interaction, then there must exist an upper limit of velocity of the signal propagation. Then the hyperbolic one – the Lorentz transformation is the only choice.

11.5 light velocity invariance

In the above discussion, c behaves like the upper limit of velocity, which is a frame independent constant. This is consistent with the relativity principle, since if c is different in different frames, then we could distinguish different frames according to the value of c, which is against the relativity principle.

Another statement is that the velocity c in one frame should also be transformed to c in another frame, denoted as $f(c) = c$. If it is not the case, say, $f(c) = c' < c$, then $c = f^{-1}(c')$. Since c is the maximum of f^{-1}, consider a small change $c - \Delta v$, there should exist $f^{-1}(c' + \Delta v_1) = f^{-1}(c' - \Delta v_2) = c - \Delta c$. It means that $f(c - \Delta v)$ would have two different values, i.e., a velocity of $c - \Delta v$ in one frame would become undetermined in another frame, which is unacceptable.

We can also easily figure out the law of superposition of velocities. If F_1 is moving at u_1 relative to F_0, and a particle is moving at u_2 relative to F_1. A co-moving frame F_2 with that particle can be set, i.e., the particle is at rest in F_2. The parameter angles corresponding to $u_{1,2}$ can be defined as

$$\tanh \theta_{1,2} = u_{1,2}/c. \tag{11.33}$$

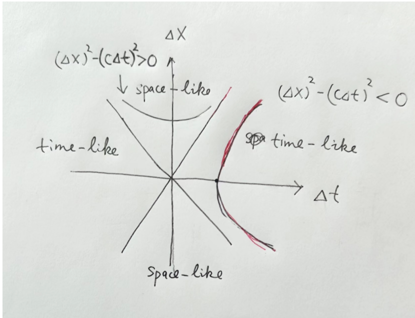

Figure 11.3 Three kinds of space-time intervals between two events according to $(\Delta x)^2 - (c\Delta t)^2 > 0$, < 0, and $= 0$, respectively. They are denoted as space-like, time-like, and light-like, respectively. For a space-like interval, Δx does not change sign but Δt can, hence, there is no causality between two events. In contrast, causality only exists for a time-like interval, for which Δt does not change sign but Δx does not.

Then the transformation of F_2 relative to F_0 is parameterized by the angle of $\theta = \theta_1 + \theta_2$. Then the velocity that F_2 relative to F_0 is

$$\tanh\theta = \frac{\tanh\theta_1 + \tanh\theta_2}{1 + \tanh\theta_1 \tanh\theta_2}$$
$$u = \frac{u_1 + u_2}{1 + \frac{u_1 u_2}{c^2}}. \tag{11.34}$$

It is also easy to prove that the Lorentz transformation maintains $(\Delta x)^2 - (c\Delta t)^2 = C$ where C is a constant. Depending $C > 0$, $C < 0$, and $C = 0$, we can divide three classes of space-time intervals as space-like, time-like, and light-like. For a time-like interval, it can be proven that the sign of Δt never changes under Lorentz transformations, which means causality is maintained. For a space-like interval, the sign of Δx never changes, but the sign Δt can change, which means that there is no causality between them.