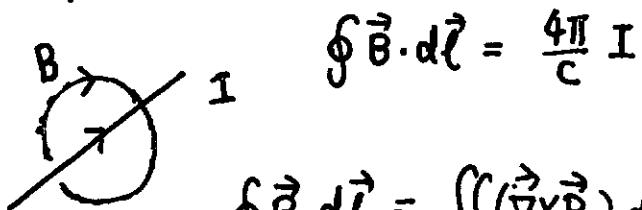


# Lect:2: Magnetic fields from steady currents

## { Ampere's law



$$\oint \vec{B} \cdot d\vec{l} = \frac{4\pi}{c} I$$

we start from here to derive  
Biot-Savart's law

$$\oint \vec{B} \cdot d\vec{l} = \iint (\nabla \times \vec{B}) d\vec{S} = \frac{4\pi}{c} \iint \vec{j} \cdot d\vec{S}$$

$$\Rightarrow \boxed{\nabla \times \vec{B} = \frac{4\pi}{c} \vec{j}}$$

clearly Ampere's law only applies  
to steady-current

$$\nabla \cdot (\nabla \times \vec{B}) = \frac{4\pi}{c} \nabla \cdot \vec{j} = 0 \Rightarrow \frac{\partial p}{\partial t} = 0.$$

we will consider  
time-dependent  
case later.

Suppose that we know the distribution

of  $\vec{j}(x, y, z)$ , but it's not enough. We need to the divergence  
of  $\vec{B}$ . The fact is that so far no magnetic monopole is discovered.

Any closed surface, the magnetic flux is zero  $\oint \vec{B} \cdot d\vec{S} = 0 \Rightarrow \boxed{\nabla \cdot \vec{B} = 0}$ .

## { Vector potential

$$\nabla \cdot \vec{B} = 0 \Rightarrow \vec{B} = \nabla \times \vec{A}, \quad \boxed{\vec{A} \text{ is well-defined up to } \vec{A} + \nabla f}$$

$$\nabla \times \vec{B} = \nabla \times (\nabla \times \vec{A}) = \frac{4\pi}{c} \vec{j}$$

$$\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} = \frac{4\pi}{c} \vec{j}$$

we further impose the condition  $\nabla \cdot \vec{A} = 0 \Rightarrow$

$$\boxed{\begin{aligned} \nabla^2 \vec{A} &= -\frac{4\pi}{c} \vec{j} \\ \nabla \cdot \vec{A} &= 0 \end{aligned}}$$

Assuming  $\vec{j}$  goes to zero at infinity, we can read off from the solution of Poisson equation:

$$\vec{A}(\vec{r}) = \frac{1}{c} \iiint \frac{\vec{j}(\vec{r}') d^3 r'}{|\vec{r} - \vec{r}'|}$$

$$\begin{aligned} \text{check } \nabla \cdot \vec{A}(\vec{r}) &= 0. \Rightarrow \nabla \cdot \vec{A}(\vec{r}) = \frac{1}{c} \iiint \vec{j}(\vec{r}') \cdot \nabla_{\vec{r}} \frac{1}{|\vec{r} - \vec{r}'|} d^3 \vec{r}' \\ &= -\frac{1}{c} \iiint \vec{j}(\vec{r}') \cdot \nabla_{\vec{r}'} \frac{1}{|\vec{r} - \vec{r}'|} d^3 \vec{r}' \\ &= -\frac{1}{c} \iiint \left[ \nabla_{\vec{r}'} \left[ \frac{\vec{j}(\vec{r}')}{|\vec{r} - \vec{r}'|} \right] - (\nabla' \vec{j}(\vec{r}')) \frac{1}{|\vec{r} - \vec{r}'|} \right] d^3 \vec{r}' \end{aligned}$$

$$\nabla \cdot \vec{A}(\vec{r}) = -\frac{1}{c} \iint \frac{\vec{j}(\vec{r}')}{|\vec{r} - \vec{r}'|} \cdot dS' = 0 \leftarrow \boxed{\begin{array}{l} \text{assuming } j(r) \rightarrow 0 \\ \text{at } r \rightarrow +\infty, \text{ or boundary} \end{array}}$$

Example: 2D, E and B duality:

Assume an 2D distribution  $j_z(x, y)$ , which does not depend on  $z$ . We cannot use the above formula because,  $j_z$  extends to  $z = \pm\infty$ .

~~B~~ only has in-plane component. define  $\vec{E} = \vec{B} \times \hat{z}$ , and  $\vec{A} = \varphi(x, y) \hat{z}$

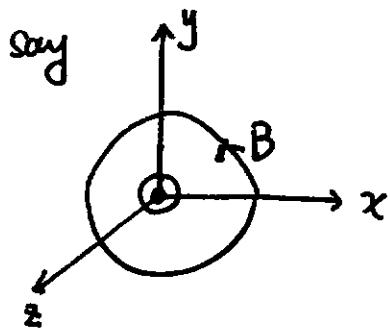
$$\Rightarrow \nabla \times \vec{A} = \nabla \times (\varphi(x, y) \hat{z}) = (\nabla \cdot \varphi) \times \hat{z} = \vec{B}$$

$$[(\nabla \cdot \varphi) \times \hat{z}] \times \hat{z} = \vec{B} \times \hat{z}$$

$$-\nabla \varphi = \vec{E} = \vec{B} \times \hat{z}$$

$$\Rightarrow -\nabla^2 \varphi = \nabla \cdot (\vec{B} \times \hat{z}) = \hat{z} \cdot \nabla \times \vec{B} = j_z(x, y)$$

so we transform it to an electro-static problem.



$$\vec{B} = \frac{2I}{Cr} e_\phi = \frac{2I}{C} \left( \frac{-y\hat{x} + x\hat{y}}{x^2 + y^2} \right)$$

$$j_z = \delta(r)$$

$$\Rightarrow \vec{E} = \vec{B} \times \vec{z} = \frac{2I}{C} \frac{\hat{r}}{r} \Rightarrow -\nabla \phi = \frac{2I}{C} \frac{\hat{r}}{r}$$

$$\Phi(r) - \Phi_{(a)} = \frac{-2I}{C} \ln r \Big|_a^r \Rightarrow \Phi(r) = -\frac{2I}{C} \ln r = -\frac{I}{C} \ln \left( \frac{x^2 + y^2}{a^2} \right)$$

↖ short range cut off

### § Magnetic fields from a general shape of wire

$$J = \frac{I}{a} \quad a: \text{cross section of wire}$$

$$d^3 r' = adl$$

$$\Rightarrow \vec{J} d^3 r' = I d\vec{l}$$

$$\Rightarrow A(\vec{r}) = \frac{I}{C} \int \frac{d\vec{l}}{|\vec{r} - \vec{r}'|}, \text{ we will find here} \Rightarrow \text{Biot-Savart law.}$$



$$dA(\vec{r}) = \frac{I}{C} \frac{d\vec{l}}{|\vec{r} - \vec{r}'|} \quad \text{thus the contribution to } \vec{B} \text{ from this line segment is}$$

$$dB(r) = \nabla \times dA(\vec{r}) = \frac{I}{C} \nabla \times \left( \frac{d\vec{l}}{|\vec{r} - \vec{r}'|} \right) = \frac{I}{C} \left( -d\vec{l} \times \nabla \frac{1}{|\vec{r} - \vec{r}'|} \right)$$

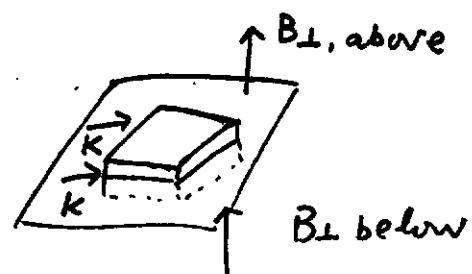
$d\vec{l}$  is a const vector respect to  $\nabla_r$

$$\begin{aligned} \Rightarrow d\vec{B}(\vec{r}) &= -\frac{I}{C} d\vec{l} \times \nabla_r \frac{1}{|\vec{r} - \vec{r}'|} = -\frac{I}{C} d\vec{l} \times \left( -\frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \right) \\ &= \frac{I}{Cr^2} d\vec{l} \times \hat{r}, \quad \text{where } \hat{r} = \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|}. \end{aligned}$$

$$\vec{B}(\vec{r}) = \frac{I}{C} \oint \frac{d\vec{l} \times \hat{r}}{r^2}$$

## § magneto-static boundary condition

When there's surface current  $K$ , magnetic fields can be discontinuous.



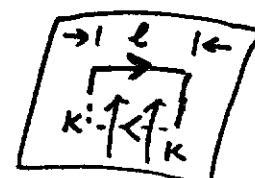
But the norm component remains continuous.

$$\oint \vec{B} \cdot d\vec{S} = 0 \Rightarrow (B_{\perp, \text{above}} - B_{\perp, \text{below}}) \cdot S = 0 \Rightarrow B_{\perp, \text{above}} = B_{\perp, \text{below}}$$

$$\oint \vec{B} \cdot d\vec{l} = \frac{4\pi}{c} \vec{K} \cdot d\vec{l}$$

$$\Rightarrow (\vec{B}_{\text{above}}'' - \vec{B}_{\text{below}}'') \cdot d\vec{l} = \frac{4\pi}{c} \vec{K} \cdot (\hat{n} \times d\vec{l})$$

$$\Rightarrow \vec{B}_{\text{above}}'' - \vec{B}_{\text{below}}'' = \frac{4\pi}{c} \vec{K} \times \hat{n}$$



$$= \frac{4\pi}{c} d\vec{l} \cdot (\vec{K} \times \hat{n})$$

We may also need boundary conditions for vector potential  $A$ .

Since  $A$  satisfies 2nd differential  $\nabla^2 A = 0$ , its discontinuity appears at its derivative.  $\vec{A}$  itself is continuous.

$\nabla \cdot \vec{A} = 0 \Rightarrow$  the normal component of  $A$  is continuous

$\oint \vec{A} \cdot d\vec{l} = \oint B \cdot (dl \cdot dh) \rightarrow 0$  as the thickness of  $dh \rightarrow 0$   
B is regular.

Set the triad frame  $\hat{n}$ ,  $\hat{k}$ , and  $\hat{k} \times \hat{n}$ .

$\vec{B} \cdot \hat{k} \times \hat{n} = -\partial_n A_k + \partial_k A_n$ , only this component of  $B$  is discontinuous

we don't expect discontinuity of  $\partial_k A_n$ , because  $A_n$  is continuous and  $\hat{k}$  is parallel to the boundary. The discontinuity comes from  $\partial_n A_k$ .

$$\Rightarrow (\partial_n A_k)_{\text{above}} - (\partial_n A_k)_{\text{below}} = -\frac{4\pi}{c} K$$

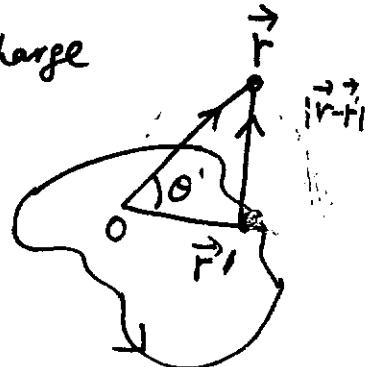
$$\text{or } (\partial_n \vec{A})_{\text{above}} - (\partial_n \vec{A})_{\text{below}} = -\frac{4\pi}{c} \vec{K}$$

in comparison: boundary condition for electric surface charge

$$(\vec{E}_{||})_{\text{above}} = (\vec{E}_{||})_{\text{below}}$$

$$(\vec{E} \cdot \hat{n})_{\text{above}} - (\vec{E} \cdot \hat{n})_{\text{below}} = 4\pi \sigma$$

$$\text{or } -\left(\frac{\partial \phi}{\partial n}\right)_{\text{above}} + \left(\frac{\partial \phi}{\partial n}\right)_{\text{below}} = 4\pi \sigma$$



### § Multiple expansion of the vector potential

we need to use

$$\frac{1}{|r-r'|} = \frac{1}{\sqrt{r^2 + r'^2 - 2rr' \cos\theta'}} = \frac{1}{r} \sum_{n=0}^{\infty} \left(\frac{r'}{r}\right)^n P_n(\cos\theta')$$

$$\vec{A}(r) = \frac{I}{c} \oint \frac{d\vec{l}'}{|r-\vec{r}'|} = \frac{I}{c} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \oint (r')^n P_n(\cos\theta') d\vec{l}'$$

$$= \frac{I}{c} \left[ \underbrace{\frac{1}{r} \oint d\vec{l}'}_{\text{monopole}} + \underbrace{\frac{1}{r^2} \oint r' \cos\theta' d\vec{l}'}_{\text{dipole}} + \underbrace{\frac{1}{r^3} \oint r'^2 \left( \frac{3}{2} \cos^2\theta' - \frac{1}{2} \right) d\vec{l}'}_{\text{quadrupole}} + \dots \right]$$

The magnetic dipole component

$$\vec{A}_{\text{dip}}(r) = \frac{I}{cr^2} \oint r' \cos\theta' d\vec{l}' = \frac{I}{cr^2} \oint (\hat{r} \cdot \vec{r}') d\vec{l}'$$

using the identity  $\oint (\vec{C} \cdot \vec{r}) d\vec{l}' = \vec{a} \times \vec{C}$ , where  $\vec{C}$  is a const vector  
 $\vec{a} = \frac{1}{2} \oint \vec{r} \times d\vec{l}'$

(6)

$$\oint (\hat{r} \cdot \vec{r}') d\vec{l}' = \vec{a} \times \hat{r} = - \left[ \frac{1}{2} \oint \vec{r}' \times d\vec{l}' \right] \times \hat{r}$$

$$\Rightarrow \vec{A}_{\text{dip}}(\vec{r}) = \frac{1}{c} \frac{\vec{m} \times \hat{r}}{r^2}, \text{ where } m = \frac{1}{2} \oint \vec{r}' \times d\vec{l}' .$$

where  $\vec{a} = \oint \vec{r}' \times d\vec{l}'$  is the vector area of a surface, which is determined by the boundary. (C.f. Prob 1-61).

E field from a dipole.

$$\vec{B}_{\text{dip}}(\vec{r}) = \nabla \times \vec{A}_{\text{dip}}(\vec{r}) = \frac{1}{c} \nabla \times \left( \frac{\vec{m} \times \hat{r}}{r^2} \right)$$

$$\text{using } \nabla \times (\vec{A} \times \vec{B}) = (\vec{B} \cdot \vec{\nabla}) \vec{A} - (\vec{A} \cdot \vec{\nabla}) \vec{B} + \vec{A} (\nabla \cdot \vec{B}) - \vec{B} (\nabla \cdot \vec{A})$$

$$\nabla \times \left( \vec{m} \times \frac{\hat{r}}{r^2} \right) = -(\vec{m} \cdot \vec{\nabla}) \left( \frac{\hat{r}}{r^2} \right) + \vec{m} \left( \nabla \cdot \frac{\hat{r}}{r^2} \right) \leftarrow \boxed{\vec{m} \cdot 4\pi \delta(\vec{r})} \text{ singlear point}$$

$$\nabla(\vec{A} \cdot \vec{B}) = \vec{A} \times (\nabla \times \vec{B}) + \vec{B} \times (\nabla \times \vec{A}) + (\vec{A} \cdot \vec{\nabla}) \vec{B} + (\vec{B} \cdot \vec{\nabla}) \vec{A} \Rightarrow \text{at origin, neglected!}$$

$$\Rightarrow \nabla \left( \frac{\vec{m} \cdot \hat{r}}{r^2} \right) = \vec{m} \cdot \nabla \frac{\hat{r}}{r^2}$$

$$\Rightarrow \nabla \times \left( \vec{m} \times \frac{\hat{r}}{r^2} \right) = -\nabla \left( \frac{\vec{m} \cdot \hat{r}}{r^2} \right) = -(\vec{m} \cdot \vec{\nabla}) \nabla \frac{1}{r^2} - \nabla(\vec{m} \cdot \vec{r}) \frac{1}{r^3}$$

$$\nabla \frac{1}{r^3} = -\frac{3\hat{r}}{r^4} \quad \nabla(\vec{m} \cdot \vec{r}) = \vec{m}$$

$$\Rightarrow \nabla \times \left( \vec{m} \times \frac{\hat{r}}{r^2} \right) = \frac{3(\vec{m} \cdot \hat{r}) \hat{r}}{r^3} - \frac{\vec{m}}{r^3}$$

$$\Rightarrow \vec{B}_{\text{dip}}(\vec{r}) = \frac{1}{c} \left( \frac{3(\vec{m} \cdot \hat{r}) \hat{r} - \vec{m}}{r^3} \right)$$

The interaction energy between two magnetic dipoles

$$V = -\vec{B}_{12} \cdot \vec{m}_2 = -\frac{1}{c} \frac{1}{r_3} \left( \vec{m}_1 \cdot \vec{m}_2 - 3(\vec{m}_1 \cdot \hat{r})(\vec{m}_2 \cdot \hat{r}) \right)$$