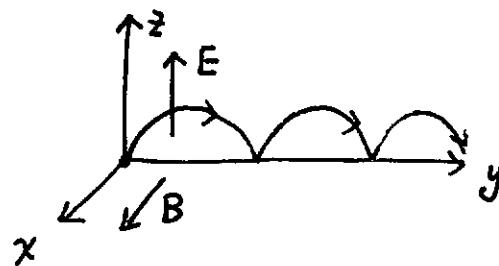


## Lect 3: Examples of magnetic fields

Griffiths Ex 5.2:

related to the classic picture  
of quantum Hall edge states



Solution: no force in the x-direction. The motion is in the yz-plane.

$$\vec{r}(t) = (0, y(t), z(t))$$

$$\vec{F}_L = q \frac{\vec{v}}{c} \times \vec{B} = \frac{qB}{c} (\hat{z}\hat{y} - \hat{y}\hat{z}), \quad \vec{F}_E = qE\hat{z}$$

$$\Rightarrow q\left(E - \frac{B}{c}\dot{y}\right)\hat{z} + \frac{qB}{c}\dot{z}\hat{y} = m\ddot{y}\hat{y} + \ddot{z}\hat{z}$$

$$\Rightarrow \frac{qB\dot{z}}{c} = m\ddot{y} \quad \text{define } \omega = \frac{qB}{mc}, \leftarrow \text{cyclotron frequency}$$

$$\left\{ \begin{array}{l} qE - \frac{qB}{c}\dot{y} = m\ddot{z} \end{array} \right.$$

$$\Rightarrow \left\{ \begin{array}{l} \dot{y} = \omega\dot{z} \\ \ddot{z} = \omega\left(\frac{EC}{B} - \dot{y}\right) \end{array} \right.$$

$$\Rightarrow \ddot{z} = -\omega\ddot{y} = -\omega^2\dot{z}$$

$$\Rightarrow \dot{z} = A\omega s\omega t + B\sin\omega t$$

$$\Rightarrow z = \frac{C_1\omega s\omega t + C_2\sin\omega t + C_3}{\omega^2[C_1\omega s\omega t + C_2\sin\omega t]}$$

$$\ddot{z} = -\omega^2[C_1\omega s\omega t + C_2\sin\omega t]$$

$$\Rightarrow \dot{y} = -\frac{\ddot{z}}{\omega} + \frac{EC}{B} = \omega[C_1\omega s\omega t + C_2\sin\omega t] + \frac{EC}{B}$$

$$\Rightarrow y = +C_1\sin\omega t - C_2\omega s\omega t + \frac{EC}{B}t + C_4$$

plug into the initial condition  $y(0) = z(0) = \dot{y}(0) = \dot{z}(0) = 0$

$$\Rightarrow \left\{ \begin{array}{l} y(t) = \frac{EC}{\omega B}(\omega t - \sin\omega t) \\ z(t) = \frac{EC}{\omega B}(1 - \cos\omega t) \end{array} \right. \quad \text{define } R = \frac{EC}{B\omega} \text{ radius}$$

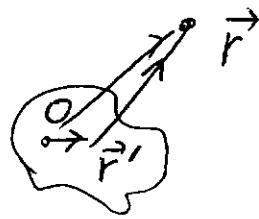
$$\Rightarrow (y - R \frac{\omega t}{\omega})^2 + (z - R)^2 = R^2$$

Cycloid

$$\boxed{v = \frac{EC}{B}}$$

\* read Page 209 - 211.

2: Start from Biot-Savart law to prove  $\nabla \cdot \vec{B} = 0$ .



$$\vec{B}(r) = \frac{1}{c} \int \frac{\vec{J}(r') \times \hat{r}_{12}}{r_{12}^2} d^3 r'$$

$$\nabla \cdot \vec{B}(r) = \frac{1}{c} \int \nabla_r \cdot \left[ \frac{\vec{J}(r') \times (\vec{r}_0 - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \right] d^3 r'$$

$$= \frac{1}{c} \int \vec{J}(r') \cdot \left[ \nabla_r \times \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \right] d^3 r' = 0$$

$$\nabla \times \vec{B}(r) = \frac{1}{c} \int \nabla_r \times \left[ \frac{\vec{J}(r') \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \right] d^3 r'$$

$$\nabla \times \left( \vec{J}(r') \times \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \right) = \vec{J}(r') \left( \nabla_r \cdot \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \right) - \left( \vec{J}(r') \cdot \nabla_r \right) \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3}$$

The second term

$$- \vec{J}(r') \cdot \nabla_r \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} = \left( \vec{J}(r') \cdot \nabla_r \right) \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} = \underbrace{\nabla_r \cdot \left( \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3}, \vec{J}(r') \right)}_{\text{derivative}} \xrightarrow{\text{total}}$$

~~using  $\nabla \cdot (\vec{A} \cdot \vec{B}) = \vec{B} \cdot \nabla \vec{A} + \vec{A} \cdot \nabla \vec{B}$~~

$$- \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \cdot \nabla_r \vec{J}(r') \xrightarrow{0}$$

⇒ the second term vanishes after volume integral

$$\Rightarrow \nabla \times \vec{B}(r) = \frac{1}{c} \int \vec{J}(r') \cdot \nabla_r \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} d^3 r' = \frac{4\pi}{c} \int \vec{J}(r') \delta(\vec{r} - \vec{r}') d^3 r'$$

$$= \frac{4\pi}{c} \vec{J}(r)$$

### 3: Application of Biot-Savart law

§ B-field from a long straight line:

$d\vec{l}' \times \hat{p}$  points out of page, with the

$$\text{magnitude } dl' \sin \alpha = dl' \cos \theta. \quad l' = r \cot \theta \Rightarrow dl' = r \sec^2 \theta d\theta$$

$$P^2 = \frac{r^2}{\cos^2 \theta} \Rightarrow B = \frac{I}{c} \int \cdot \frac{d\vec{l}' \times \hat{p}}{P^2}$$

$$B = \frac{I}{c} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{r \sec^2 \theta \cos \theta d\theta}{r^2 / \cos^2 \theta} = \frac{I}{cr} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos \theta d\theta = \frac{I}{cr} \left[ \sin \theta \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{2I}{cr}$$

The force between two parallel wire  $\vec{F} = \int (\vec{B} \times \vec{B}) dq = \int (\vec{B} \times \vec{B}) \lambda dl$

$$\Rightarrow \vec{F} = \int (\vec{B} \times \vec{B}) \frac{dl}{c} \Rightarrow \boxed{\frac{F}{l} = \frac{2I_1 I_2}{c^2 d}}$$

§: B-field at distance  $z$  above the center of a circular loop of radius  $R$  with a steady current  $I$ .

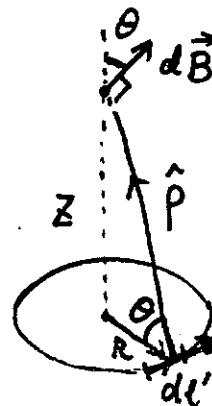
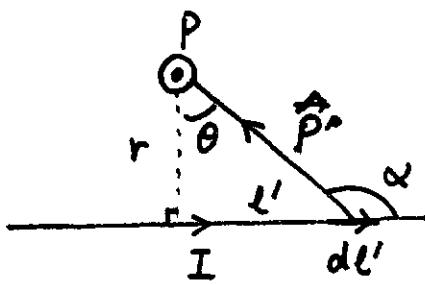
$d\vec{l}' \times \hat{p}$  has the magnitude  $dl'$ , it's direction

as plotted forming a polar angle  $\theta$  with the  $z$ -axis.

$$\Rightarrow B_z = \frac{I}{c} \int \frac{dl' \cos \theta}{P^2} = \frac{I \cos \theta}{P^2 c} \cdot 2\pi R = \frac{2\pi I}{c} \frac{R^2}{P^3}$$

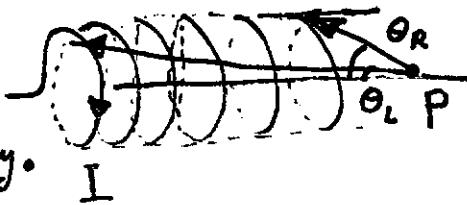
$$= \frac{2\pi I}{c} \frac{R^2}{(R^2 + z^2)^{3/2}} \quad \text{or} \quad \boxed{B_z = \frac{2\pi I}{c} \frac{\cos^3 \theta}{R}}$$

Other components average to zero.



## § B-field at axis of a wound solenoid

Suppose that the left and right ends span the polar angles of  $\theta_L$  and  $\theta_R$ , respectively.



$$dB = \frac{2\pi\lambda dl}{CR} \sin^3 \theta = +\frac{2\pi\lambda}{CR} \sin \theta d\theta$$



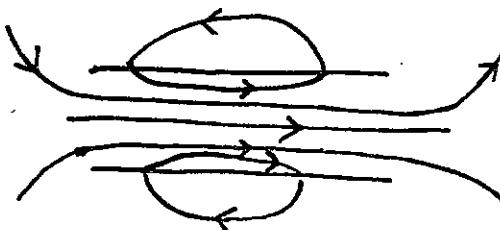
$$B = \int_{\theta_L}^{\theta_R} dB = -\frac{2\pi\lambda}{CR} R \cos \theta \Big|_{\theta_L}^{\theta_R} = \frac{2\pi\lambda}{C} [ws\theta_L - ws\theta_R] \quad l = R \cot \theta$$

$$dl = +R \csc^2 \theta d\theta$$

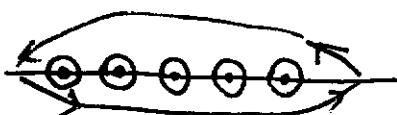
for an infinitely long solenoid,

$$\theta_L = 0, \theta_R = \pi \Rightarrow B = \frac{4\pi\lambda}{C},$$

where  $\lambda = I \cdot n$ , and  $n$  is the num of turns per length.



B-field of a solenoid.



## 4: Application of Ampere's law + symmetric analysis

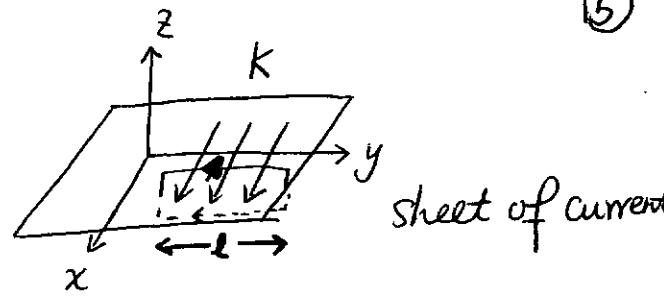
① B can only along "circumferential" direction.



$$\oint B \cdot dl = \frac{4\pi I}{c} \Rightarrow B \cdot 2\pi r = \frac{4\pi I}{c} \Rightarrow B = \frac{2I}{cr}$$

\* B-field from a sheet-current.

B should have translational symmetry, i.e. B is uniform along xy-direction.



① Can B have a z-component? Combined

No. The system has the symmetry of time-reversal and rotation along z-axis 180°.

This operation flip the direction of  $B_z$ . So B can only be in the plane.

② The system has the symmetry of time-reversal and reflectional respect to  $\bar{z}y$  plane.

$B$  is an axial vector  $\Rightarrow B_x \xrightarrow{\text{com}} -B_x \xrightarrow{\text{TR}} -B_x \xrightarrow{\text{Ref}} -B_x$ . Thus  $B_x = 0$ .

③ B can only along y-direction.  $B_y$  should not depend on  $z$ , for  $z > 0$

Let us choose a loop at  $z > 0 \Rightarrow B_y(z) = B_y(z + \Delta z)$ .

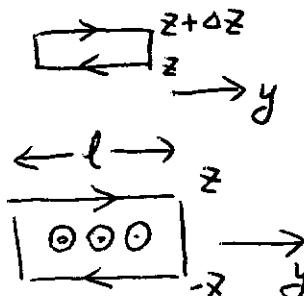
for a loop crossing the current sheet.

$B_y \cdot 2l$

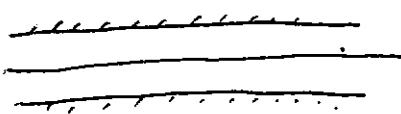
The system has rotation symmetry around x-axis at 180°

$$\Rightarrow B_y(z) = -B_y(-z) \Rightarrow B_y(z) \cdot 2 \cdot l = -\frac{4\pi}{c} K \cdot l$$

$$\Rightarrow B_y(z) = \begin{cases} -\frac{2\pi}{c} K & \text{for } z > 0 \\ \frac{2\pi}{c} K & \text{for } z < 0 \end{cases}$$



\* B-field from an infinitely long solenoid



The system has rotational symmetry around the axis.

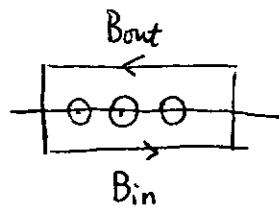
① B cannot have radial component, otherwise  $\oint B \cdot ds \neq 0$ .

(6)

②  $\vec{B}$  cannot have "circumferential" component, otherwise  $\oint \vec{B} \cdot d\vec{l} \neq 0$ .

③  $\vec{B}$  can only be along axial. It can also be proved that  $\vec{B}$  is uniform inside the solenoid, and outside.

$$(\vec{B}_{in} - \vec{B}_{out}) \cdot \vec{l} = \frac{4\pi}{c} I N$$



$B_{out} = 0$  if we set  $r \rightarrow \infty$

$$\Rightarrow \vec{B}_{in} = \frac{4\pi}{c} I n \text{ along axis.}$$

\*: a toroidal coil of a circular ring (actually can be any shape). The winding is uniform. What's the distribution of  $\vec{B}$ -field?

we first use Biot-Savart law to prove that  $\vec{B}$  is only circumferential. This can also be proved simply by symmetry.

Our system has rotational symmetry around "z"-axis, without loss of generality, let us consider a point  $r$  in the  $xz$ -plane, with  $\vec{r} = (x, 0, z)$

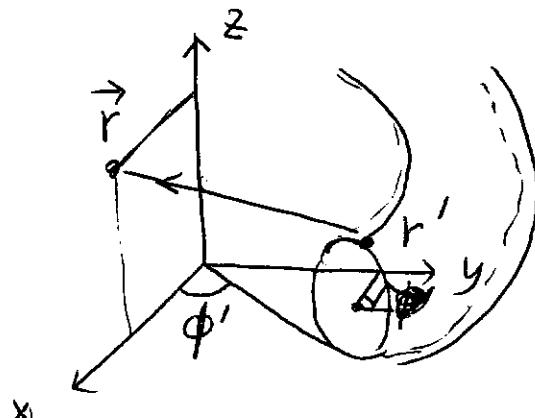
the coordinate of  $r'$  on the toroidal coil

$$\vec{r}' = (s' \cos \phi', s' \sin \phi', z') \quad \text{azimuthal angle.}$$

the current  $I$  has no  $\phi$  dependence

$$\vec{I} = (I_s \cos \phi', I_s \sin \phi', I_z)$$

$$\Rightarrow d\vec{B} = \frac{1}{c} \frac{\vec{I} \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} d\ell'$$



$$\vec{I} \times (\vec{r} - \vec{r}') = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ I_s \cos\phi' & I_s \sin\phi' & I_z \\ x - S' \cos\phi' & -S' \sin\phi' & z - z' \end{vmatrix} = \sin\phi' (I_s(z-z') + S'I_z) \hat{x} \\ + [I_z(x - S'\cos\phi') - I_s \cos\phi'(z-z')] \hat{y} \\ + [-I_s x \sin\phi'] \hat{z}$$

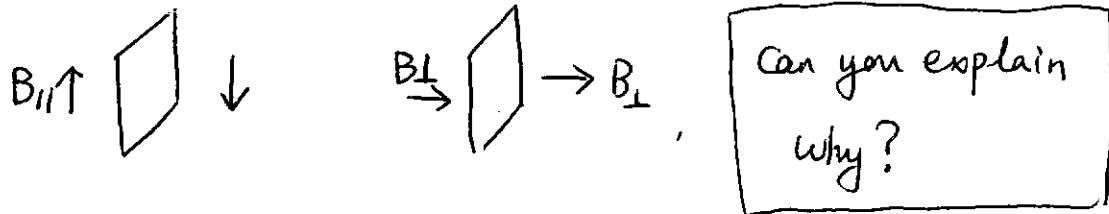
vanishes

the contribution to  $\hat{x}$  and  $\hat{z}$  are odd functions of  $\phi' \Rightarrow$  after integration

$\Rightarrow d\vec{B}$  only along the  $\hat{y}$ -direction, or  $\vec{B}$  is along "circumferential".

Or we can simply get it from symmetry analysis.

$\vec{B}$  is axial-vector. It has different properties under reflection operation.



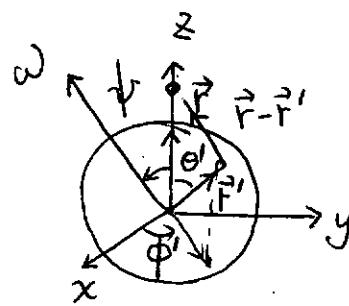
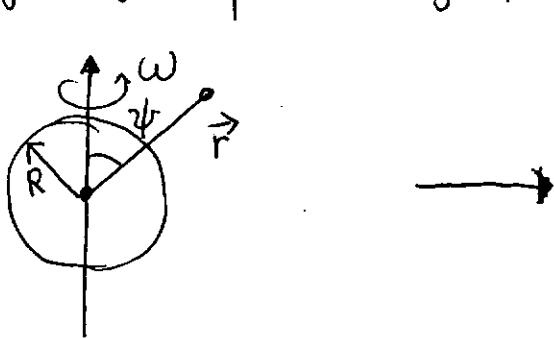
- ① ~~our system has rotation symmetry~~ radial
- ② our system has reflection symmetry respect to any vertical plane  $\Rightarrow B$  cannot parallel to that plane.  $B$  can only perpendicular to the vertical-radial plane.  
 $\Rightarrow B$  is circumferential.

Then the results are straight-forward. For point  $\vec{r}$  inside the torus

~~$$\vec{B}(p) \cdot 2\pi p = \frac{4\pi}{C} I \cdot N \rightarrow B(p) = \frac{2IN}{pc}$$
 for  $p$  in  $p$  to the  $z$ -axis with radius~~

$$B(r) = \frac{2I}{C} \frac{N}{r}, \quad \text{otherwise } B(r) = 0.$$

# S Magnetic field of a rotating spherical shell



we rotate  $\vec{r}$  to the  $z$ -axis, and  $\vec{\omega}$  in the  $x$ - $z$  plane.

$\vec{r}'$  is on the sphere with  $(\theta', \phi')$

$$\vec{A}(\vec{r}) = \frac{1}{c} \int \frac{\vec{K}(r')}{|\vec{r}-\vec{r}'|} da' \quad \text{where } \vec{K}(r') = \sigma \vec{v}$$

$$|\vec{r}-\vec{r}'| = \sqrt{R^2 + r^2 - 2Rr \cos \theta'}$$

$$da' = R^2 \sin \theta' d\theta' d\phi'$$

$$\vec{v} = \vec{\omega} \times \vec{r}' = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ w \sin \psi & 0 & w \cos \psi \\ R \sin \theta' \cos \phi' & R \sin \theta' \sin \phi' & R \cos \theta' \end{vmatrix}$$

$$= R\omega [-\cos \psi \sin \theta' \sin \phi' \hat{x} + (\cos \psi \sin \theta' \cos \phi' - \sin \psi \cos \theta') \hat{y} + \sin \psi \sin \theta' \sin \phi' \hat{z}]$$

those terms contains  $\cos \phi'$  &  $\sin \phi'$  go to zero after average over  $\phi'$

$\Rightarrow$  only  $A_y(\vec{r}) \neq 0$ .  $-\hat{y}$  is the direction of  $\vec{\omega} \times \vec{r}$ .

$$\Rightarrow \vec{A}(\vec{r}) = \frac{R^2 \omega}{c} \int_0^{2\pi} \frac{\sin \psi \cos \theta' \sin \theta'}{\sqrt{R^2 + r^2 - 2Rr \cos \theta'}} d\theta' = \frac{2\pi R^2 \omega}{c} \int_0^{\pi} \frac{\sin \psi \cos \theta' d\theta' \sin \theta'}{\sqrt{R^2 + r^2 - 2Rr \cos \theta'}}$$

$$\int_{-1}^1 \frac{u du}{\sqrt{R^2 + r^2 - 2Rr u}} = - \frac{R^2 + r^2 + Rr u}{3R^2 r^2} \sqrt{R^2 + r^2 - 2Rr u} \Big|_{-1}^{+1} =$$

$$= -\frac{1}{3R^2 r^2} [(R^2 + r^2 + Rr) |R-r| - (R^2 + r^2 - Rr)(R+r)]$$

$$= \begin{cases} \frac{2r}{3R^2} & (R > r) \\ \frac{2R}{3r^2} & (R < r) \end{cases}$$

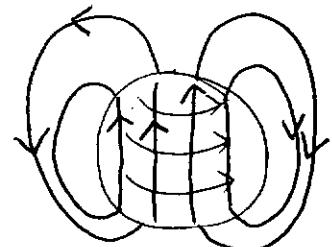
remember  $\vec{\omega} \times \vec{r} = -\omega r \sin \psi \hat{y}$

⑨

$$\Rightarrow \vec{A}(r) = \begin{cases} \frac{4\pi}{3c} R \sigma (\vec{\omega} \times \vec{r}) & \text{if } r \text{ is inside sphere} \\ \frac{4\pi}{3c} \frac{R^4}{r^3} \sigma (\vec{\omega} \times \vec{r}) & \text{if } r \text{ is outside sphere} \end{cases}$$

if set back  $\vec{\omega}$  along  $z$ -axis  $\Rightarrow$

$$A(r, \theta, \phi) = \begin{cases} \frac{4\pi}{3c} R \omega \sigma r \sin\theta \hat{e}_\phi & (r \leq R) \\ \frac{4\pi}{3c} \frac{R^4}{r^2} \omega \sigma \sin\theta \hat{e}_\phi & (r \geq R) \end{cases}$$



$$\vec{B} = \nabla \times \vec{A}(r)$$

$$\nabla \times (\vec{\omega} \times \vec{r}) = -(\underbrace{\vec{\omega} \cdot \vec{r})}_{\vec{\omega}} \hat{r} + \vec{\omega} (\nabla \cdot \hat{r}) = \vec{\omega}$$

$$\Rightarrow \vec{B}_{\text{inside}} = \frac{8\pi}{3c} R \sigma \vec{\omega}$$

$$\nabla \times (\vec{\omega} \times \frac{\hat{r}}{r^2}) = -(\vec{\omega} \cdot \vec{r})(\frac{\hat{r}}{r^2}) + \vec{\omega} (\nabla \cdot \frac{\hat{r}}{r^2})$$

$$= -\nabla \left( \frac{\vec{\omega} \cdot \hat{r}}{r^2} \right) + \vec{\omega} \underbrace{4\pi \delta(\vec{r})}_{\text{go to zero, because } |\vec{r}| > R}$$

$$\nabla \left( \frac{\vec{\omega} \cdot \hat{r}}{r^3} \right) = -(\vec{\omega} \cdot \vec{r}) \nabla \frac{1}{r^3} - \vec{\omega} \nabla \left( \frac{1}{r^3} \right)$$

$$\nabla \times (\vec{\omega} \times \frac{\hat{r}}{r^2}) = \frac{3(\vec{\omega} \cdot \hat{r}) \hat{r}}{r^3} - \frac{\vec{\omega}}{r^3}$$

$$\vec{B}_{\text{outside}} = \frac{4\pi}{3c} R^4 \sigma \left[ \frac{3(\vec{\omega} \cdot \hat{r}) \hat{r} - \vec{\omega}}{r^3} \right] \quad \text{its a dipolar field}$$

with  $\boxed{\vec{m} = \frac{4\pi}{3c} R^4 \sigma \vec{\omega}}$