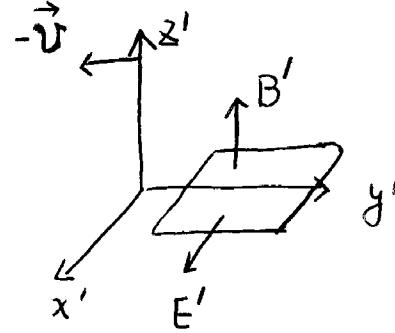
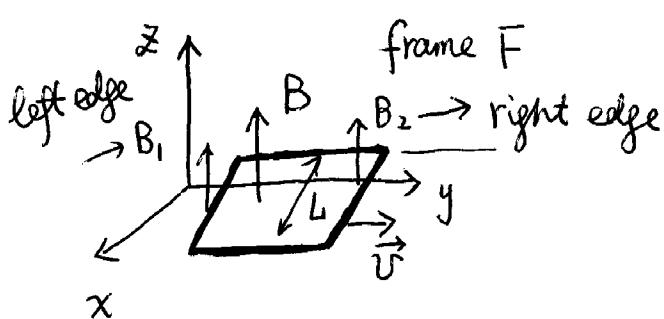


## Lect 5: Electro-magnetic induction

{ a loop of wire moving in non-uniform B-field



lab frame with zero E field.

Two edges (left and right) will feel Lorentz force, and the other edges, along the loop

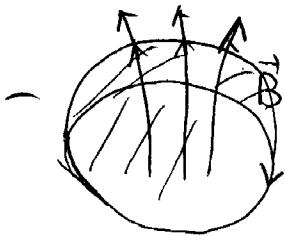
the Lorentz force is perpendicular to the loop.

$$\oint \vec{f} \cdot d\vec{s} = \frac{qV}{c} (B_2 - B_1) L \quad \text{define emf: } \mathcal{E} = \frac{\oint \vec{f} \cdot d\vec{s}}{q} = \frac{LV}{c} (B_2 - B_1)$$

or we define magnetic flux  $\Phi = \iint d\vec{a} \cdot \vec{B}$

$$\frac{d\Phi}{dt} = \frac{B_2 L V \sin t - B_1 L V \sin t}{\Delta t} = (B_2 - B_1) L V \Rightarrow \boxed{\mathcal{E} = -\frac{1}{c} \frac{d\Phi}{dt}}$$

due to the fact that there's no magnetic charge  $\nabla \cdot \vec{B} = 0$ , the flux penetrating a surface only depends on the boundary. As long as the boundary is specified, it doesn't matter which surfaces you measure.



$\mathcal{E}$  can also be represented as

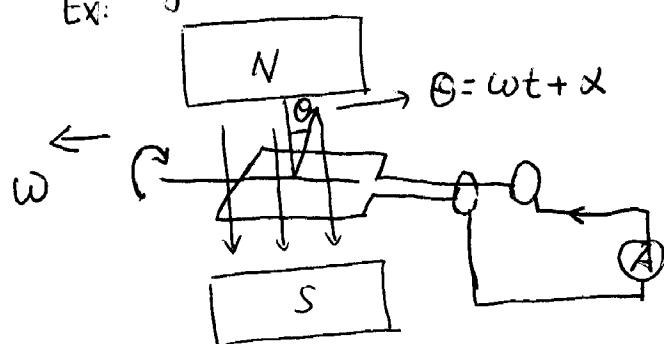
$$\mathcal{E} = \frac{1}{c} \oint (\vec{v} \times \vec{B}) \cdot d\vec{l}$$

$$\mathcal{E} = -\frac{1}{c} \frac{d\Phi}{dt} \quad \leftarrow \text{Lenz's law.}$$

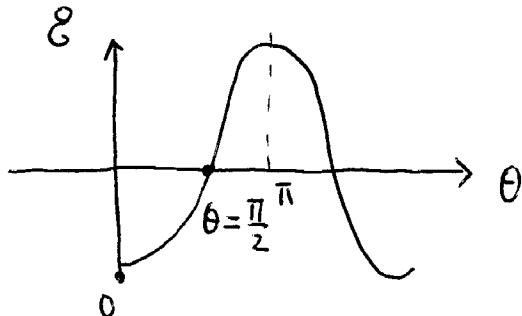
(2)

generated  
the flux by the induced  
current should be in an opposite  
direction to the change of  $\Phi$ .  
(not  $\Phi$  itself, but  $d\Phi$ !)

Ex: generator



$$\Phi = S \cdot B \sin(\omega t + \alpha) \Rightarrow \mathcal{E} = -\frac{1}{c} \frac{d\Phi}{dt} = -\frac{SB\omega}{c} \cos(\omega t + \alpha)$$



§: a stationary loop in a changing B-field

In the comoving frame, there exists electric field  $E'$ . The emf is purely generated by  $E'$ .  $\vec{E}' = -\frac{\vec{v}' \times \vec{B}'}{c} = \frac{\vec{v} \times \vec{B}'}{c}$

$$\oint \vec{E}' \cdot d\vec{s}' = \frac{L v}{c} (\vec{B}'_1 - \vec{B}'_2) \quad \text{again} \quad \mathcal{E}' = -\frac{1}{c} \frac{d\Phi'}{dt'}$$

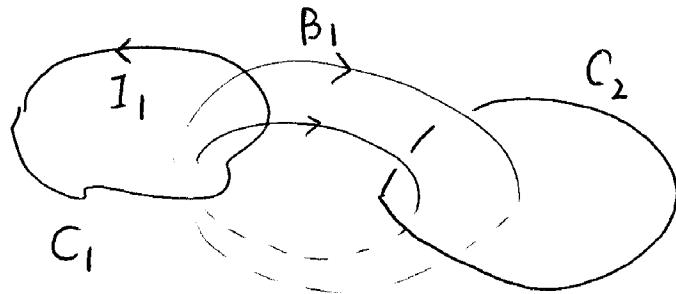
please note that we need to use  $\mathcal{E}'$ ,  $B'$  and  $t'$  consistently.

→ Faraday's law  $\oint \vec{E} \cdot d\vec{s} = -\frac{1}{c} \frac{d}{dt} \iint \vec{B} \cdot d\vec{a}$

$$\Leftrightarrow \nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}$$

## § mutual conductance

Consider two current loops  $C_1$  and  $C_2$  (position fixed,



flux from  $I_1$  through loop  $C_2$

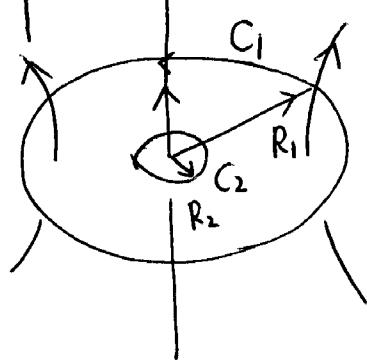
$$\Phi_{21} = \iint_{S_2} \vec{B} \cdot d\vec{a}_2 = \propto I_1 \cdot \text{Const}$$

$$\Rightarrow E_{21} = - \frac{\text{Const}}{C} \cdot \frac{dI_1}{dt} = - M_{21} \frac{dI_1}{dt}$$

Similarly we can calculate the emf in loop 1, generated by the current change of  $C_2$ .

$$E_{12} = - M_{12} \frac{dI_2}{dt}$$

Ex:  $B_1$  at the center of the ring



$$B_1 = \frac{2\pi I_1}{CR_1}, \quad \Phi_{21} = \pi R_2^2 \frac{2\pi I_1}{CR_1} = \frac{2\pi^2 I_1 R_2^2}{CR_1}$$

$$\Rightarrow E_1 = - \frac{2\pi^2 R_2^2}{C^2 R_1} \frac{dI_1}{dt},$$

$$M_{21} = \boxed{\frac{2\pi^2 R_2^2}{C^2 R_1}}$$

$$[M] = \text{Henry} = \frac{[\text{Volt}]}{[\text{Amp}] \cdot [\text{s}]}$$

parameters of  $R_1, R_2$   
are not symmetric

reciprocal theorem:  $\boxed{M_{21} = M_{12}}$ , why?

$$\left. \begin{aligned} - \Phi_{21} &= \oint_{C_2} \vec{A}_{21}(\vec{r}_2) \cdot d\vec{l}_2 \\ \vec{A}_{21}(\vec{r}_2) &\equiv \oint_{C_1} \frac{I_1(\vec{r}_1) d\vec{l}_1}{r_{21}} \end{aligned} \right\} \Rightarrow \Phi_{21} = \frac{I_1}{C} \oint_{C_1} \oint_{C_2} \frac{d\vec{l}_2 \cdot d\vec{l}_1}{r_{21}}$$

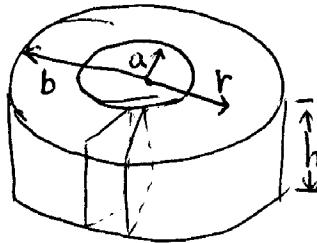
$$\Rightarrow M_{21} = \frac{1}{C} \oint_{C_2} \oint_{C_1} \frac{d\vec{l}_2 \cdot d\vec{l}_1}{r_{21}} = M_{12}$$

(4)

§ Self-inductance — the emf in loop C<sub>1</sub> generated by the change of current I,

$$\mathcal{E}_{11} = -\frac{1}{C} \frac{d\Phi_{11}}{dt}$$

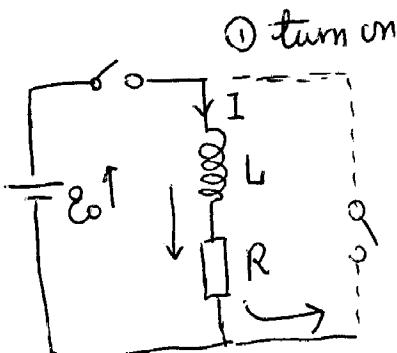
$$= -L_1 \frac{dI_1}{dt}$$



$$\Rightarrow B \cdot 2\pi r = \frac{4\pi}{C} NI \Rightarrow B = \frac{2N}{Cr} I$$

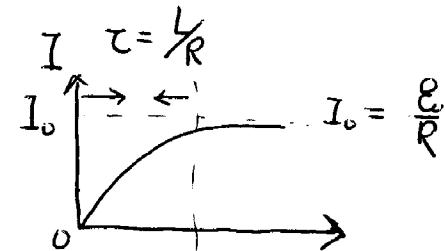
$$\Phi = \underbrace{N}_{h} \int_a^b \frac{2NI}{Cr} dr = \frac{2NIh}{C} \ln\left(\frac{b}{a}\right) \Rightarrow L = \frac{2N^2 h}{C^2} \ln\left(\frac{b}{a}\right)$$

§ circuit contains R and L



$$u = IR = \mathcal{E}_0 - L \frac{dI}{dt}$$

$$\text{or } \left\{ \begin{array}{l} \frac{dI}{dt} = -\frac{R}{L} I + \frac{\mathcal{E}_0}{L} \\ I(0) = 0 \end{array} \right.$$

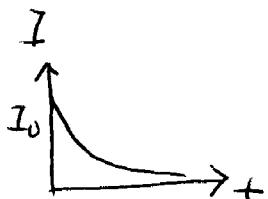


$$\Rightarrow I - \frac{\mathcal{E}_0}{R} = -\frac{\mathcal{E}_0}{R} e^{-\frac{R}{L}t} \Rightarrow I = \frac{\mathcal{E}_0}{R} \left( 1 - e^{-\frac{R}{L}t} \right)$$

L prevent the jump of current.

② turn off

$$-L \frac{dI}{dt} = RI \Rightarrow I = I_0 e^{-R/L t}$$



§ Energy stored in magnetic field

The energy dissipation on R

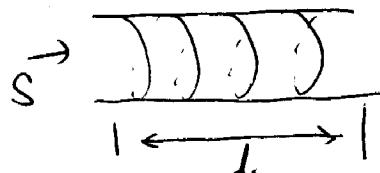
$$U = R \int_0^{+\infty} I^2(t) dt = R I_0^2 \int_0^{+\infty} e^{-\frac{2R}{L}t} dt = R I_0^2 \cdot \frac{L}{2R} = \frac{LI_0^2}{2}$$

This amount of energy was originally stored in the coil as magnetic

- energy

$$U = \frac{1}{2} L I^2$$

Consider a long coil



$$\Rightarrow U = \frac{1}{2} \frac{4\pi}{c} \frac{N^2 S}{l} I^2$$

$$= \left( \frac{4\pi}{c} \frac{NI}{l} \right)^2 \frac{S \cdot l}{8\pi}, = \frac{B^2 \cdot \text{Vol}}{8\pi}$$

$$\Rightarrow \text{energy (magnetic) density} = \frac{B^2}{8\pi} \quad \text{c.f. electric energy density} \frac{E^2}{8\pi}$$

- generalize to ⑩ non-uniform E and B field

$$U = \frac{1}{8\pi} \int (E^2 + B^2) dV$$

$$\Phi = NSB = NSB$$

$$B = \frac{4\pi}{c} \frac{NI}{l}$$

$$\Rightarrow \Phi = \frac{4\pi}{c} \frac{N^2 SI}{l}$$

$$\Rightarrow L = \frac{4\pi}{c} \frac{N^2 S}{l}$$