

## Lect 6. Displacement current

what's left?

$$\nabla \cdot \vec{E} = 4\pi \rho \quad \text{Gauss's law}$$

$$\nabla \cdot (\nabla \times \vec{B}) = \frac{4\pi}{c} \nabla \cdot \vec{j} \neq 0 \quad \nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t} \quad \text{Faraday's law}$$

$$\nabla \cdot \vec{B} = 0 \quad \text{no magnetic monopole}$$

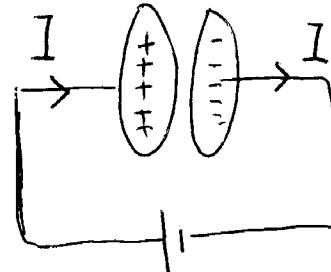
$$\nabla \times \vec{B} = \frac{4\pi}{c} \vec{j} \quad \text{Ampere's law}$$

$$\nabla \cdot \vec{j} + \frac{\partial \rho}{\partial t} = 0 \quad \text{continuity equation}$$

$$\nabla \cdot (\vec{j} + \vec{j}_D) = 0 \Rightarrow \nabla \cdot \vec{j}_D = + \frac{\partial \rho}{\partial t} = + \frac{1}{4\pi c} \nabla \cdot \vec{E} \Leftarrow \text{choose } \vec{j}_D = \frac{1}{4\pi} \frac{\partial}{\partial t} \vec{E}$$

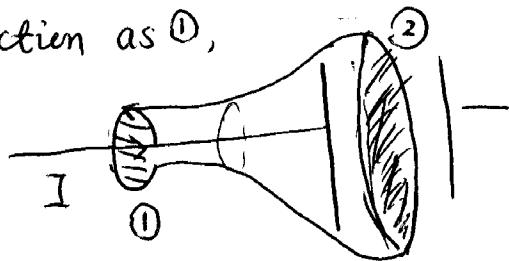
$$\Rightarrow \boxed{\nabla \times \vec{B} = \frac{4\pi}{c} \vec{j} + \frac{1}{c} \frac{\partial}{\partial t} \vec{E}} \quad \text{Maxwell's contribution}$$

Suppose we provide a steady charge current to a capacitor.



we know  $\oint \vec{B} \cdot d\vec{l} = \frac{4\pi}{c} I$  if I choose

the cross section as ①,



I can also choose the cross section as ②.

there are no I.  $\oint \vec{B} \cdot d\vec{l} = 0$  ? No! We can pretend that

current is continuous, but this time it's relate by  $j_D = \frac{1}{4\pi} \frac{\partial E}{\partial t}$

$$\text{we can show that } E = 4\pi \sigma \Rightarrow j_D = \frac{1}{4\pi} \frac{\partial E}{\partial t} = \frac{\partial \sigma}{\partial t} = \frac{I}{S} = j$$

## § E & M waves

In the region without charge and current, Maxwell equations

reduces to

$$\left\{ \begin{array}{l} \nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}, \quad \nabla \cdot \vec{E} = 0 \\ \nabla \times \vec{B} = \frac{1}{c} \frac{\partial \vec{E}}{\partial t}, \quad \nabla \cdot \vec{B} = 0 \end{array} \right.$$

E.M field can

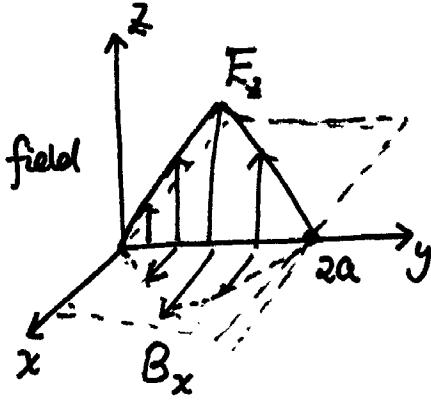
in the free space without source. It does not propagate

need a media like sound wave.  $\vec{E}, \vec{B}$  do not need an underlying material field like in the elasticity theory.

Consider at  $t=0$ , in the region between

$y=0$  and  $2a$ , we have the following  $\vec{E}, \vec{B}$  field

$$(t=0) \left\{ \begin{array}{l} E_z = E_0 \frac{y}{a}, \quad (0 \leq y \leq a) \\ E_z = E_0 \left( \frac{2a-y}{a} \right), \quad (a \leq y \leq 2a) \\ B_x = B_0 \frac{y}{a}, \quad (0 \leq y \leq a) \\ B_x = B_0 \left( \frac{2a-y}{a} \right) \quad (a \leq y \leq 2a) \end{array} \right.$$



they satisfy  $\vec{B} = \hat{y} \times \vec{E}$

we can make it propagates by plugging in  $y \rightarrow y - ct$ , i.e.

$$\left\{ \begin{array}{l} E_z = E_0 \frac{y-ct}{a} \quad 0 \leq y-ct \leq a \\ E_z = E_0 \frac{2a-(y-ct)}{a} \quad a \leq y-ct \leq 2a \end{array} \right.$$

and  $\left\{ \begin{array}{l} B_x = B_0 \frac{y-ct}{a} \quad 0 \leq y-ct \leq a \\ B_x = B_0 \left( \frac{2a-(y-ct)}{a} \right) \quad a \leq y-ct \leq 2a \end{array} \right.$

Check in the region  $0 \leq y-ct \leq a$

$$\nabla \times \vec{E} = \hat{x} \frac{\partial E_z}{\partial y} = \frac{E_0}{a} \hat{x}, \quad \nabla \times \vec{B} = -\hat{z} \frac{\partial B_x}{\partial y} = -\frac{B_0}{a} \hat{z}$$

$$\nabla \cdot \vec{E} = 0$$

$$\nabla \cdot \vec{B} = 0$$

in the region  $a \leq y - ct \leq 2a$

$$-\nabla \times \vec{E} = -\frac{E_0}{a} \hat{x}, \quad \nabla \times \vec{B} = \frac{B_0}{a} \hat{z}$$

Similarly, in the region  $0 \leq y - ct \leq a$  and  $a \leq y - ct \leq 2a$

$$\frac{\partial E}{\partial t} = -\frac{c}{a} E_0 \hat{z}, \quad \frac{\partial B}{\partial t} = -\frac{c}{a} B_0 \hat{x}; \quad \frac{\partial E}{\partial t} = \frac{c}{a} E_0 \hat{z}, \quad \frac{\partial B}{\partial t} = \frac{c}{a} B_0 \hat{x}$$

$\Rightarrow$  they satisfy  $\nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}$ , and  $\nabla \times \vec{B} = \frac{1}{c} \frac{\partial \vec{E}}{\partial t}$ .

§ In general: for a  $\vec{E}, \vec{B}$  configuration  
and propagation direction  $\hat{k}$

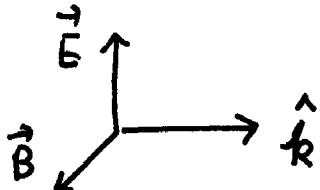
form a triad, with  $\boxed{\vec{B}(\vec{r} \cdot \hat{k} - ct) = \hat{k} \times \vec{E}(\vec{r} \cdot \hat{k} - ct)},$

- it satisfies Maxwell equation.

$$\nabla \times \vec{B} = \nabla \times (\hat{k} \times \vec{E}(\vec{r} \cdot \hat{k} - ct)) = \hat{k} (\nabla \cdot \vec{E}) - (\hat{k} \cdot \vec{\nabla}) \vec{E}$$

$$= -(\hat{k} \cdot \vec{\nabla}) \vec{E} = -\frac{1}{c} \frac{\partial}{\partial t} \vec{E}$$

$$\nabla \times \vec{E} = \nabla \times (-\hat{k} \times \vec{B}(\vec{r} \cdot \hat{k} - ct)) = -\hat{k} (\nabla \cdot \vec{B}) + (\hat{k} \cdot \vec{\nabla}) \vec{B} = \frac{1}{c} \frac{\partial}{\partial t} \vec{B}$$



From Lorentz transformation of  $\vec{E}, \vec{B}$  field, we can check

$E^2 - B^2$  and  $\vec{E} \cdot \vec{B}$  are invariant.

$\Rightarrow |E| = |B|$  and  $\vec{E} \perp \vec{B}$  cannot change.

light velocity is always "C" in any frame.

suppose in the frame F, the  $\vec{E}, \vec{B}$  fields satisfy

$$\begin{cases} \vec{B} = f(x - ct) \hat{z} \\ \vec{E} = f(x - ct) \hat{y} \end{cases}$$

In the frame  $F'$ , in which  $E', B'$  transforms

$$B'_z(x', t') = \gamma(B_z(x, t) - \beta E_y(x, t)) = \gamma(1-\beta) f(x-ct)$$

$$E'_y(x', t') = \gamma(E_y(x, t) - \beta B_z(x, t)) = \gamma(1-\beta) f(x-ct)$$

$$x = \gamma x' - \gamma \beta c t' \Rightarrow x - ct = \gamma(1+\beta)(x' - c't')$$

$$ct = -\gamma \beta x' + \gamma c t'$$

$$\Rightarrow \begin{cases} B'_z(x', t') = \gamma(1-\beta) f(\gamma(1+\beta)(x' - c't')) = \gamma(1-\beta) f'(x' - ct') \\ E'_y(x', t') = \gamma(1-\beta) f(\gamma(1+\beta)(x' - c't')) = \gamma(1-\beta) f'(x' - ct') \end{cases}$$

$\Rightarrow B'_z, E'_y$  satisfy the Maxwell equation

$$\nabla' \times \vec{B}' = \frac{1}{c} \frac{\partial \vec{E}'}{\partial t'}, \quad \nabla' \times \vec{E}' = -\frac{1}{c} \frac{\partial \vec{B}'}{\partial t'}$$

