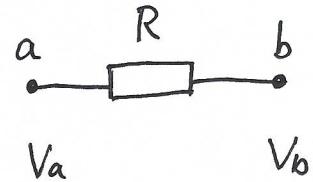


# AC circuits and beyond

## { Impedance and admittance (lumped elements)

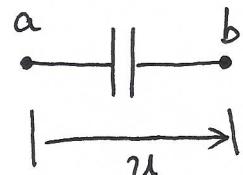
- Resistance



$$u(t) = V_a - V_b = U_0 e^{i\omega t}$$

$$I(t) = u(t)/R = I_0 e^{i\omega t} \quad \text{with } I_0 = U_0/R, \text{ or } \frac{U_0}{I_0} = R$$

- Capacitance

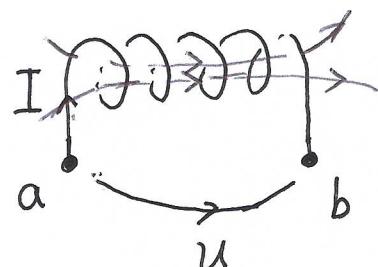


$$u(t) = Q(t)/C \Rightarrow \dot{u} = I(t)/C$$

$$i\omega U_0 = I_0/C \Rightarrow \frac{U_0}{I_0} = \frac{1}{i\omega C}$$

- Inductance

$$\oint \vec{E} \cdot d\vec{s} = \int_a^b \vec{E} \cdot d\vec{s}_{\text{coil}} + \int_b^a \vec{E} \cdot d\vec{s}$$



$$= \mathcal{E} = -L \frac{d\Phi}{dt}$$

Inside the coil, there's no E field in metal  $\Rightarrow \int_a^b \vec{E} \cdot d\vec{s}_{\text{coil}} = 0$

$$\Rightarrow u(t) = V_a - V_b = - \int_b^a \vec{E} \cdot d\vec{l} = -\mathcal{E} = L \frac{dI}{dt}$$

$$U_0 = L I_0 i\omega \Rightarrow \boxed{\frac{U_0}{I_0} = i\omega L}$$

Now we can define impedance  $Z = R + i\omega L + \frac{1}{i\omega C}$  and

$$I_o = U_o/Z.$$

- EMF (electric motive force).

Consider a loop. The total force that a charge  $q$  feels,  $\vec{F} = q\vec{E} + \vec{F}_{\text{ot}}$

$\oint \vec{F}_g \cdot d\vec{l} = \mathcal{E}$ , where  $\mathcal{E}$  is the electric motive force. is denoted as

Certainly, for a purely electrostatic system,  $\vec{F} = -q \nabla \varphi$ , then

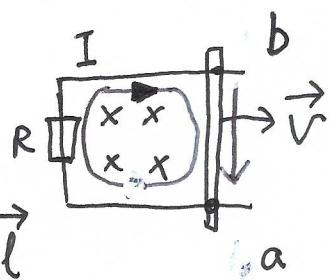
$\mathcal{E} = - \oint \nabla \phi \cdot d\vec{l} = 0$ . Hence, if  $\mathcal{E} \neq 0$ , there must be something beyond electrostatics.

**Case 1:** Faraday's law  $\oint \vec{E}_q \cdot d\vec{l} = \oint \vec{E} \cdot d\vec{l} = -\frac{1}{c} \frac{d\Phi}{dt} = 8$

Case 2: Lorentz force

$$= \oint_{\text{II}} \vec{E} \cdot d\vec{l} + \int_b^a \frac{\vec{v} \times \vec{B}}{c} \cdot d\vec{l},$$

Since  $\vec{E}$  is electrostatic



3

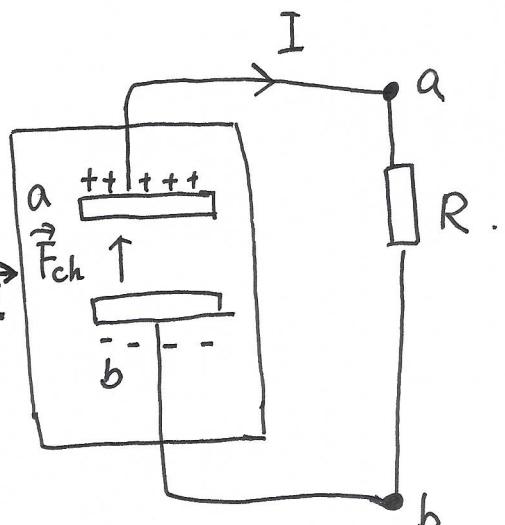
On the other hand, inside metal  $\vec{E} + \frac{\vec{v} \times \vec{B}}{c} = 0$ ,

Everything is consistent!

- ### • Case 3 chemical battery

$$Q = \oint \vec{F}_q \cdot d\vec{l} = \int_a^b \vec{E} \cdot d\vec{l} + \int_b^a (\vec{E} + \frac{\vec{F}_{ch}}{q}) \cdot d\vec{l}$$

via R                      via battery



$$= \oint \vec{E} \cdot d\vec{l} + \frac{1}{q} \int_b^a \vec{F}_{ch} \cdot d\vec{l}$$

$$\Rightarrow \mathcal{E} = \frac{1}{2} \int_b^a \vec{F}_{ch} \cdot d\vec{l}$$

Inside battery,  $\vec{F}_{ch} + q\vec{E} + \vec{f} = 0$ ,  $\vec{f}$ : friction.

$$\Rightarrow V_a - V_b = - \int_b^a \vec{E} \cdot d\vec{l} = - \int_b^a \vec{E} \cdot d\vec{l}$$

via R                      via battery

$$= \int_b^a \frac{1}{q} \vec{F}_{ch} \cdot d\vec{l} + \frac{1}{q} \int_b^a \vec{f} \cdot d\vec{l} = \mathcal{E} + \frac{1}{q} \int_b^a \vec{f} \cdot d\vec{l}$$

## effect of internal resistance

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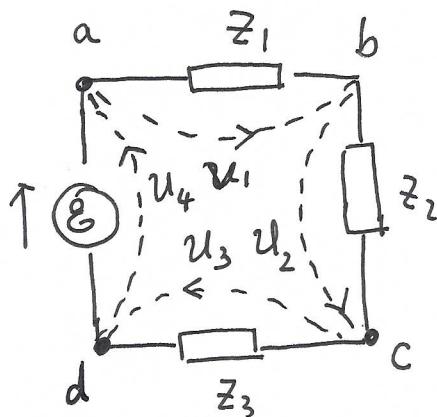
since  $\vec{f} \propto -\vec{v}$ ,  $\frac{1}{q} \int_b^a \vec{f} \cdot d\vec{l} = -\frac{b}{q^2} I$ , where  $b$  is

a constant. Then  $b/q^2$  can be viewed as internal resistance  $r$

$$\Rightarrow V_a - V_b = \mathcal{E} - Ir = IR \Rightarrow V_a - V_b = \frac{r}{R+r} \mathcal{E}$$

## \* Kirchhoff's law

① Consider a loop consisting of a generator and a few lumped elements. We assume that no magnetic field leaking outside these elements. Then



$$0 = \oint \vec{E} \cdot d\vec{l} = \int_a^b \vec{E} \cdot d\vec{l} + \int_b^c \vec{E} \cdot d\vec{l} + \int_c^d \vec{E} \cdot d\vec{l} + \int_d^a \vec{E} \cdot d\vec{l}$$

The integration path is along the dashed lines.  $\Rightarrow \sum u_i = 0$

Check the loop  $d \rightarrow$  generator  $\rightarrow a \rightarrow$  dashed line  $\rightarrow d$

$$\mathcal{E} = \int_a^a \left( \frac{1}{q} \vec{F}_{\text{other}} + \vec{E} \right) \cdot d\vec{l} + \int_a^d \vec{E} \cdot d\vec{l}$$

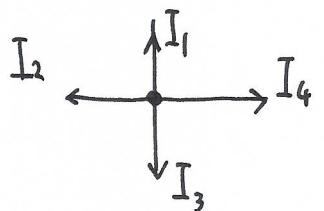
via source

if neglecting internal resistance  $\frac{\vec{F}_{\text{other}}}{q} + \vec{E} = 0$   
inside the generator

$$\Rightarrow \mathcal{E} = - \int_a^b \vec{E} \cdot d\vec{l} = -U_4$$

$$\Rightarrow U_1 + U_2 + U_3 - \mathcal{E} = 0.$$

② For each node, the total current sum must be zero.



$\sum_i I_i = 0$ , pay attention to the positive directions.

Example:

Consider the left loop:

$$U_1 + U_2 - \mathcal{E}_1 = 0$$

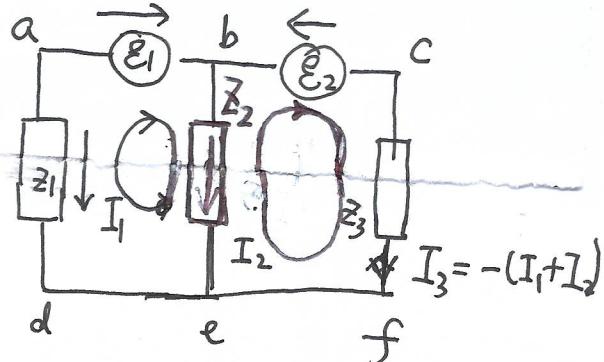
$$\Rightarrow -I_1 \cancel{\frac{Z_1}{Z_1}} + I_2 Z_2 - \mathcal{E}_1 = 0$$

$$U_2 + U_3 + \mathcal{E}_2 = 0$$

$$-I_2 Z_2 + I_3 Z_3 + \mathcal{E}_2 = 0 \Rightarrow \mathcal{E}_2 - I_2 Z_2 - (I_1 + I_2) Z_3 = 0$$

$$\Rightarrow I_1 = \frac{Z_2 \mathcal{E}_2 - (Z_2 + Z_3) \mathcal{E}_1}{Z_1 (Z_2 + Z_3) + Z_2 Z_3}$$

$$\left\{ \begin{array}{l} I_2 = \frac{Z_1 \mathcal{E}_2 + Z_3 \mathcal{E}_1}{Z_1 (Z_2 + Z_3) + Z_2 Z_3} \end{array} \right.$$



(6)

Generally speaking, Kirchhoff law applies when the wave feature of E & M is ~~very~~ prominent, i.e. when the wavelength  $\lambda \gg L$ , where L is the circuit size.

$$\text{For } \nu = 50 \text{ Hz}, \quad \lambda = c/\nu = 6 \times 10^6 \text{ m}$$

$$\nu = 100 \text{ MHz} \quad \lambda = c/\nu = 3 \text{ m}$$

$$\nu = 1 \text{ GHz} \quad \lambda = 0.3 \text{ m}$$

$$\nu = 1 \text{ THz} \quad \lambda \approx 0.3 \text{ mm}$$

$$\nu = 400 \text{ THz} \sim 800 \text{ THz} \quad \lambda \approx 400 \sim 700 \text{ nm}$$

$\swarrow$   
visible light

\* In addition, we use the approximation of lumped elements.

We neglected that circuit itself may carry inductance and have capacitance, which we call distributed impedance.

① Consider the scale of the entire circuit.

$$\oint \vec{E} \cdot d\vec{l} = -\frac{1}{c} \frac{\partial}{\partial t} \iint \vec{B} \cdot d\vec{S}, \quad \text{estimate the contribution}$$

from RHS, which is  $\sim \frac{BL^2}{CT}$  where  $T = 1/\nu$  is the period.

The RHS can be neglected if  $\frac{BL^2}{CT} \ll U$ , where  $U$  is a typical voltage drop across an element. Plug in  $B \sim \frac{I}{LC}$

$$\Rightarrow U \gg \frac{IL}{C^2 T} \iff \frac{U}{I} \cdot C \gg \frac{L}{CT}$$

$\frac{U}{I}$  is the typical resistance or impedance, whose natural unit is  $h/e^2$

$$\Rightarrow \frac{U}{I}/\frac{h}{e^2} \cdot \frac{hc}{e^2} \gg \frac{L}{CT} \quad \frac{e^2}{hc} = \alpha$$

$$\Rightarrow \boxed{Z \gg \left(\frac{L}{CT}\right) \frac{\alpha}{2\pi} \frac{h}{e^2}}$$

② The charge accumulation on a node is at the order of  $(\sum_i I_i)T$ , which can not exceed the order of  $C \cdot \mathcal{E}$

where  $C$  is the effective capacitance of the entire circuit. Since  $C$  carries the unit of length, we have

$$(\sum_i I_i) \sim \mathcal{E} \cdot L/T.$$

$\sum_i I_i$  is negligible, if  $(\sum_i I_i)/I \sim \frac{\mathcal{E}}{I} \frac{L}{T} \ll 1$ , where

$I$  is a typical value of current. It means  $Z \ll \frac{I}{L}$

$$Z \cdot (\frac{h}{e^2}) \cdot \frac{hc}{e^2} \ll \frac{TC}{L}$$

$$\Rightarrow \boxed{Z \ll \left(\frac{CT}{L}\right) \frac{\alpha}{2\pi} \frac{h}{e^2}}$$

$$\frac{h}{e^2} \simeq 26 \text{ kV} \quad \frac{\alpha}{2\pi} = \frac{1}{137 \times 6.28} = \frac{1}{860}$$

$$\frac{h}{e^2} \cdot \frac{\alpha}{2\pi} = 30\sqrt{2}.$$

Hence, the above conditions become

$$\frac{L}{CT} \ll Z/R_o \ll \frac{CT}{L}$$

\* For a daily life circuit,  $CT = 3 \times 10^8 \times 0.02 \text{ m} \simeq 6 \times 10^6 \text{ m}$

$L \sim 1 \text{ m}$ , we have  $\frac{CT}{L} \sim 6 \times 10^6$ , this should be no problem.

But if consider a power system at  $L \sim 1 \text{ km} \sim 10^3 \text{ m}$ , even

though  $\frac{CT}{L} \sim 10^4 \gg 1$ , but the typical resistance / impedance

$$10^{-4} \ll Z/R_o \ll 10^4$$

→ this is actually an important constraint.

\* For a typical radio frequency circuit,  $100 \text{ MHz}$ .  $CT \sim \frac{3 \times 10^8}{1 \times 10^8} = 3 \text{ m}$

a typical radio size  $L \sim 0.1 \text{ m} \Rightarrow \frac{CT}{L} \simeq 30$ .

The application of Kirchhoff's law requires

$$\frac{1}{30} \leq Z/R_o \ll 30$$