

AC circuits (II)

① RLC resonance

$$U_L + U_C + U_R - \mathcal{E}(t) = 0$$

$$U_L = L \frac{dI}{dt} = i\omega L I$$

$$U_C = I/(i\omega C)$$

$$U_R = IR$$

$$\Rightarrow I = \frac{\mathcal{E}_0 e^{i\omega t}}{(R + i\omega L + \frac{1}{i\omega C})}$$

The resonance occurs at $i\omega L + \frac{1}{i\omega C} = 0$, i.e. $\omega_0 = \frac{1}{\sqrt{LC}}$.

$$\text{Define } f(\omega) = \omega L - \frac{1}{\omega C} \approx \left(L + \frac{1}{\omega_0^2 C}\right)(\omega - \omega_0)$$

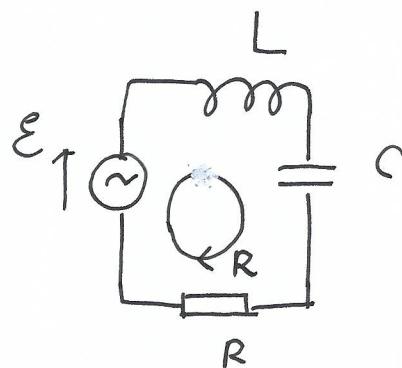
$$\approx L \left(1 + \frac{1}{\omega_0^2 C L}\right) (\omega - \omega_0) = 2L(\omega - \omega_0) \\ = 2L\omega_0 \frac{\Delta\omega}{\omega_0}$$

$$|I| = \frac{\mathcal{E}_0}{\left[R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2\right]^{1/2}} \approx \frac{\mathcal{E}_0}{\left[R^2 + \left(L\omega_0\right)^2 \left(\frac{2\Delta\omega}{\omega_0}\right)^2\right]^{1/2}}$$

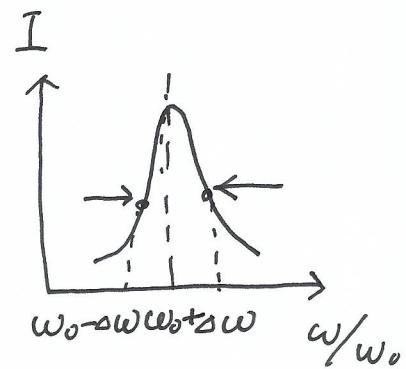
hence we define the width of resonance,

$$\frac{2|\Delta\omega|}{\omega_0} = \frac{R}{\omega_0 L} = \frac{1}{Q}$$

At $\omega = \omega_0 \pm \Delta\omega$, I falls to $\frac{1}{\sqrt{2}}$ of the peak value.



$$\mathcal{E}(t) = \mathcal{E}_0 e^{i\omega t}$$

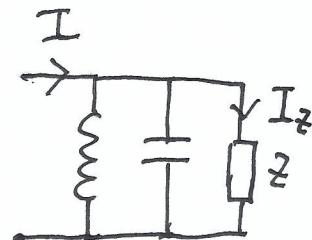


If $\mathcal{E}_{ct} = 0$, $L \frac{dI}{dt} + \frac{Q}{C} + IR = 0$

$$L \frac{d^2Q}{dt^2} + R \frac{dQ}{dt} + \frac{Q}{C} = 0 \Rightarrow \frac{1}{Z} = \frac{R}{L}, \omega_0^2 = \frac{1}{LC}$$

② parallel resonance

$$\frac{1}{Z_{LC}} \rightarrow Z_{LC} = \frac{i\omega L \cdot \frac{1}{i\omega C}}{i\omega L + \frac{1}{i\omega C}}$$



$$= \frac{\frac{1}{L/C}}{i(\omega L - 1/\omega C)}$$

$$Z_{LC}^{-1}(\omega) = \frac{i(\omega L - 1/\omega C)}{1/LC} = \frac{i}{1/LC} (L + \frac{1}{\omega_0^2 C}) \propto \omega$$

$$= iC \propto \omega$$

$$I_2/I = \frac{Y_Z}{Y_Z + Y_{LC}} = \frac{1}{1 + \frac{1}{Z_{LC}}} = \frac{1}{1 + iC \propto \omega}$$

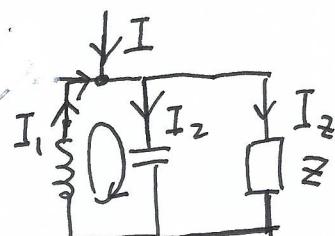
goes

① Hence, at resonance, nearly all the current goes outside

the LC loop. In fact, what really happens is that, the current

$I_1 \approx I_2$, they circulate inside

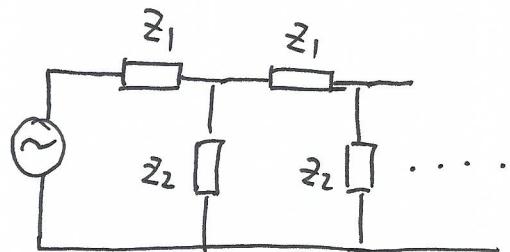
but nearly not to outside.



② if $\omega > \omega_0$, then current I mostly goes through capacitor,
if $\omega < \omega_0$, then current I goes through the inductor.

• Filter

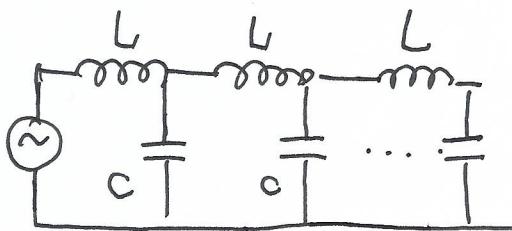
Consider a ladder circuit



Use the method of iteration. Assume that the impedance is $z_0 \Rightarrow$

$$z_0 = z_1 + \frac{z_0 z_2}{z_0 + z_2} \Rightarrow z_0^2 = z_0 z_1 + z_1 z_2 \Rightarrow z_0 = \frac{z_1}{2} + \sqrt{\left(\frac{z_1}{2}\right)^2 + z_1 z_2}$$

Now consider the case that $z_1 = \frac{i\omega L}{\text{---}}$ and $z_2 = \frac{1}{i\omega C}$



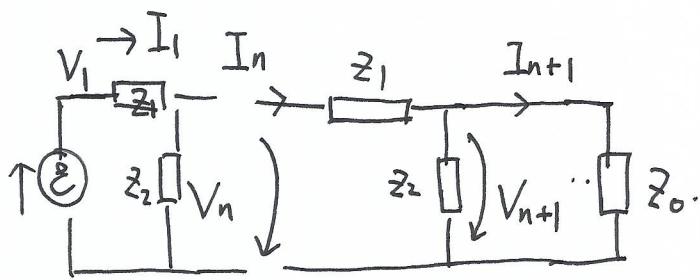
$$\begin{aligned} z_0 &= \frac{z_1}{2} \left[1 + \left(1 + \frac{4z_2}{z_1} \right)^{1/2} \right] = \frac{i\omega L}{2} + \left(\frac{L}{C} - \frac{\omega^2 L^2}{4} \right)^{1/2} \\ &= \frac{i\omega L}{2} + \frac{L}{2} \left(\frac{4}{LC} - \omega^2 \right)^{1/2} \end{aligned}$$

① if $\omega < \frac{2}{\sqrt{LC}}$, then z_0 acquires a resistance part.

How to understand it? it means electric signals propagate deeply inside the ladder to infinity.

② if $\omega > \frac{2}{\sqrt{LC}}$, $z_0 = \frac{i\omega L}{2} \left[1 + \left(1 - \frac{4}{\omega^2 LC} \right)^{1/2} \right]$, which remains inductive.

To understand what happens at $\omega > \frac{2}{\sqrt{LC}}$ and $\omega < \frac{2}{\sqrt{NLC}}$, we check. (4)



$$V_n - V_{n+1} = I_n z_1 = \frac{V_n}{z_0} z_1$$

$$\Rightarrow \frac{V_{n+1}}{V_n} = 1 - \frac{z_1}{z_0} = \alpha, \text{ where } \alpha \text{ is called the propagation factor.}$$

$$V_1 = \mathcal{E} \Rightarrow V_n = \alpha^{n-1} \mathcal{E}$$

$$\alpha = \frac{z_0 - z_1}{z_0} = \frac{\left(\frac{L}{C} - \frac{\omega^2 L^2}{4}\right)^{1/2} - i \frac{\omega L}{2}}{\left(\frac{L}{C} - \frac{\omega^2 L^2}{4}\right)^{1/2} + i \frac{\omega L}{2}}$$

$\omega < \frac{2}{\sqrt{NLC}}$

Hence in case ①, $|\alpha| = 1$, define $\alpha = e^{-i\delta}$, where

$$\tan \frac{\delta}{2} = \frac{\omega L}{2} / \left(\frac{L}{C} - \frac{\omega^2 L^2}{4} \right)^{1/2}$$

there's a phase delay when propagating into the bulk.

in case ②, $\omega > \frac{2}{\sqrt{LC}}$,

$$\alpha = \frac{\sqrt{\omega^2 L^2/4 - L/C} - \frac{\omega L}{2}}{\sqrt{\omega^2 L^2/4 - L/C} + \frac{\omega L}{2}} < 1$$

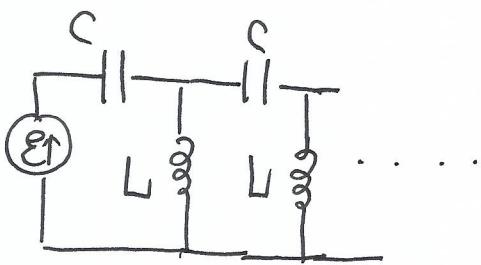
it decays as goes inside.

Hence, the system behaves like a low-pass filter. But we do not

want an infinitely long ladder, then we can use z_0 to match



how to design a high pass filter



We can base on the result for the low-pass, by mapping

$$\begin{aligned} L &\rightarrow \frac{1}{C}, \quad i\omega \rightarrow \frac{1}{i\omega} \\ C &\rightarrow \frac{1}{L} \end{aligned} \quad \left. \begin{aligned} z_0 &= \frac{1}{2i\omega C} + \frac{1}{2C} \left(4LC - \frac{1}{\omega^2} \right) \\ &= \cancel{\frac{1}{2i\omega C}} + \frac{1}{2i\omega C} + \left(\frac{L}{C} \right) \end{aligned} \right\}$$

Case I: if $\omega > \frac{1}{2\sqrt{LC}}$

$$= \frac{-i}{2\omega C} + \frac{1}{2\omega C} (4\omega^2 LC - 1)^{1/2}$$

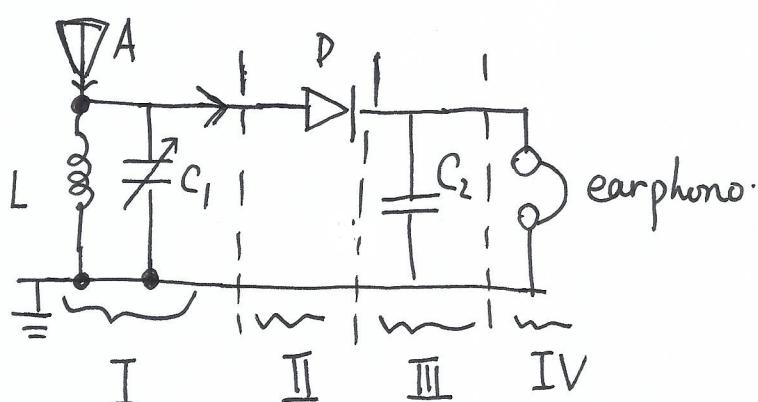
then z_0 becomes resistive, and then $|z| = 1$

Case II: if $\omega < \frac{1}{2\sqrt{LC}}$, z_0 remains conductive

and then $|z| < 1$.

* Crystal radio — no power is needed.

(6)



A: antenna

part I: tuned circuit
frequency selection

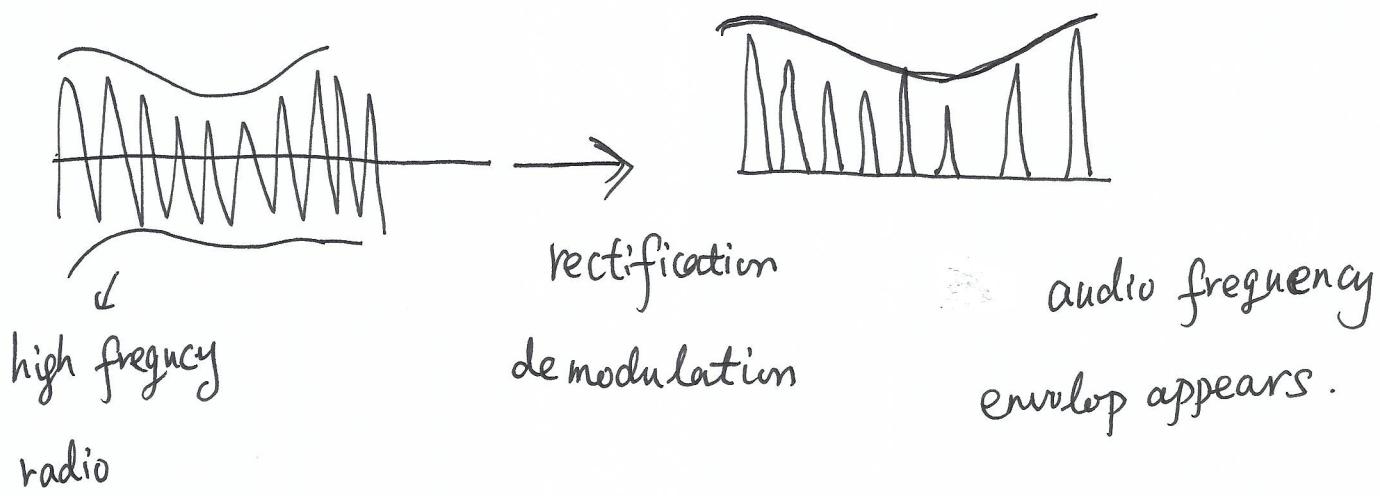
part II: demodulation

part III: filter

① Part I is a parallel resonance, the signal with $\omega \sim \omega_0 = \frac{1}{\sqrt{LC_1}}$

can move to the next step. If $\omega \ll \omega_0$, it leaks via L,
and if $\omega \gg \omega_0$, it leaks via C_1 .

② The signal input at part II is high frequency with amplitude modulation. It is rectified by the diode, typically a semi-conductor crystal detector.



part III : low pass filter to allow the audio - frequency component (audio) to pass.

part IV : earphone, low frequency signal drive earphone.