

Lect 1 Vector algebra

Vector algebra:

scalar : quantities have magnitude but no direction

mass, charge, density, temperature, etc

non-relativistic

vector : quantities have both magnitude & direction

\vec{r} , \vec{v} , $\vec{\alpha}$. Vector can be considered as an array of numbers

$$\vec{r} = (x, y, z), \quad \vec{v} = (v_x, v_y, v_z) = v_x \vec{i} + v_y \vec{j} + v_z \vec{k}$$

$$= x \vec{i} + y \vec{j} + z \vec{k}$$

↑

unit basis

$$\textcircled{1} \quad \vec{A} + \vec{B} = (A_x, A_y, A_z) + (B_x, B_y, B_z) = (A_x + B_x, A_y + B_y, A_z + B_z)$$

$$= \vec{B} + \vec{A}$$

$$\textcircled{2} \quad \vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z = \vec{B} \cdot \vec{A} \Rightarrow \vec{A} \cdot \vec{A} = A_x^2 + A_y^2 + A_z^2$$

norm $|A| = \sqrt{\vec{A} \cdot \vec{A}}$

$$\textcircled{3} \quad \vec{A} \times \vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = (A_y B_z - A_z B_y, A_z B_x - B_z A_x, A_x B_y - A_y B_x)$$

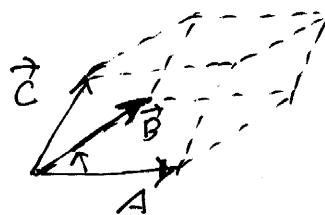
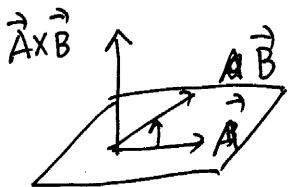
$$= -\vec{B} \times \vec{A} \Rightarrow \vec{A} \times \vec{A} = 0$$

$$\textcircled{4} \quad \vec{A} \cdot (\vec{B} \times \vec{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B})$$

$$\textcircled{5} \quad \vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$$

$$(\vec{A} \times \vec{B}) \times \vec{C} = -\vec{A}(\vec{B} \cdot \vec{C}) + \vec{B}(\vec{A} \cdot \vec{C})$$

$$\textcircled{6} \quad (\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D}) = (\vec{A} \cdot \vec{C})(\vec{B} \cdot \vec{D}) - (\vec{A} \cdot \vec{D})(\vec{B} \cdot \vec{C})$$



$$\vec{C} \cdot (\vec{A} \times \vec{B})$$

volume of the parallelepiped

{ position, displacement:

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} \quad r = \sqrt{x^2 + y^2 + z^2}$$

$$\hat{r} = \frac{\vec{r}}{r} = \frac{x\vec{i} + y\vec{j} + z\vec{k}}{\sqrt{x^2 + y^2 + z^2}}$$

$$\vec{r}_{12} = \vec{r}_1 - \vec{r}_2, \quad |\vec{r}_{12}| = |\vec{r} - \vec{r}'|, \quad \hat{r}_{12} = \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|}$$

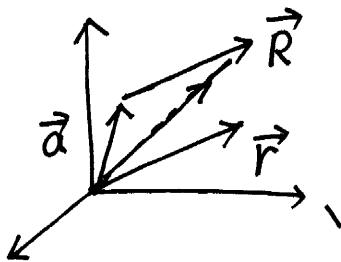
{ transformations of vector

translation

①. vector \vec{r} is translated

by \vec{a} , the resultant

$$\vec{R} = \vec{r} + \vec{a}. \quad \text{using the matrix method}$$



$$\begin{pmatrix} R_x \\ R_y \\ R_z \end{pmatrix} = \begin{pmatrix} r_x \\ r_y \\ r_z \end{pmatrix} + \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}$$

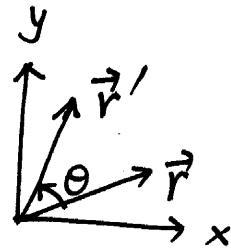
② rotation:

vector \vec{r} is rotated around the z-axis

by an angle θ :

$\vec{r}' = R_z(\theta) \vec{r}$, its components satisfy the matrix relation:

$$\begin{pmatrix} r'_x \\ r'_y \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} r_x \\ r_y \\ 0 \end{pmatrix}$$



generally, for a 3d rotation: $\vec{r}' = R \vec{r}$,

or

$$\begin{pmatrix} r'_x \\ r'_y \\ r'_z \end{pmatrix} = \begin{pmatrix} R_{xx} & R_{xy} & R_{xz} \\ R_{yx} & R_{yy} & R_{yz} \\ R_{zx} & R_{zy} & R_{zz} \end{pmatrix} \begin{pmatrix} r_x \\ r_y \\ r_z \end{pmatrix}$$

↳ 3x3 orthogonal matrix: $R^T R = I$.

$$\det R = 1$$

③ inversion

$$\vec{r}' = -\vec{r}$$

$$\text{or } \begin{pmatrix} r'_x \\ r'_y \\ r'_z \end{pmatrix} = \begin{pmatrix} -1 & & \\ & -1 & \\ & & -1 \end{pmatrix} \begin{pmatrix} r_x \\ r_y \\ r_z \end{pmatrix}$$

④ reflection with respect to xy, yz, zx-planes.

$$\text{xy: } \begin{pmatrix} r'_x \\ r'_y \\ r'_z \end{pmatrix} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix} \begin{pmatrix} r_x \\ r_y \\ r_z \end{pmatrix},$$

$$yz: \begin{pmatrix} r'_x \\ r'_y \\ r'_z \end{pmatrix} = \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} r_x \\ r_y \\ r_z \end{pmatrix}$$

$$zx: \begin{pmatrix} r'_x \\ r'_y \\ r'_z \end{pmatrix} = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} r_x \\ r_y \\ r_z \end{pmatrix}$$

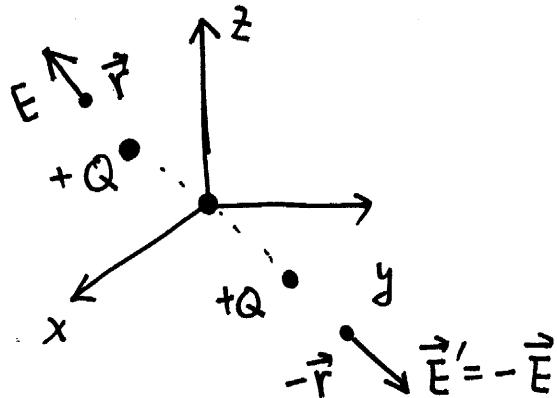
All the transformations which keep the norm of \vec{r} unchanged, can be decomposed into a series of the above operations.

3 pseudo-vector:

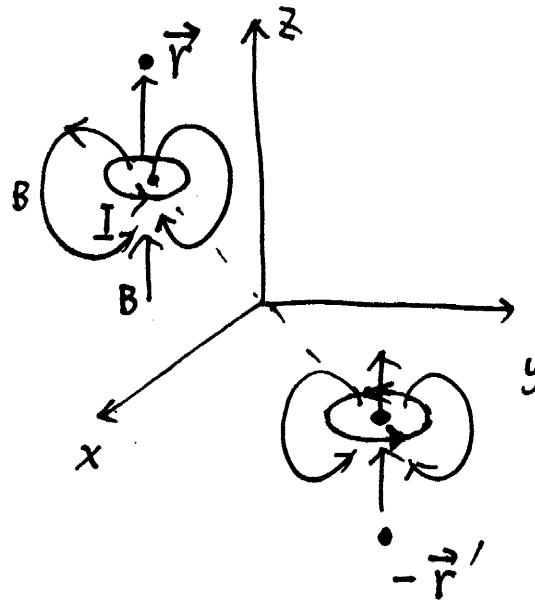
\vec{E} - vector is an ordinary vector, satisfying the above transformation.

\vec{B} - is pseudo vector. Under spatial rotations, \vec{B} transforms as the same as \vec{E} .

But under ~~to~~ inversion: $\vec{E} \rightarrow -\vec{E}$, $\vec{B} \rightarrow \vec{B}$.



inversion of a charge Q and its electric field



(5)

the inversion
of a current loop
and its surrounding
B-field.

\vec{B} is even under spatial inversion, thus is
quite different from usual vectors. — pseudo-vector
or axial vector

Lect 2 Differential calculus

§ gradient:

Consider a scalar function $T(x, y, z)$.

$$\begin{aligned} dT(x, y, z) &= \frac{\partial T}{\partial x} dx + \frac{\partial T}{\partial y} dy + \frac{\partial T}{\partial z} dz \\ &= \left(\frac{\partial T}{\partial x} \hat{x} + \frac{\partial T}{\partial y} \hat{y} + \frac{\partial T}{\partial z} \hat{z} \right) \cdot (dx \hat{x} + dy \hat{y} + dz \hat{z}) \\ &= \nabla T \cdot d\vec{r}, \text{ where } \nabla T = \frac{\partial T}{\partial x} \hat{x} + \frac{\partial T}{\partial y} \hat{y} + \frac{\partial T}{\partial z} \hat{z} \end{aligned}$$

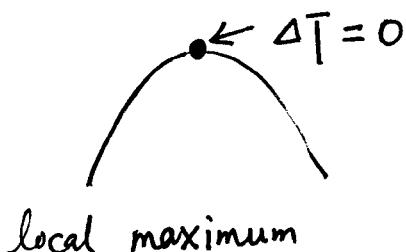
$\Rightarrow dT = |\nabla T| \cdot |d\vec{r}| \cos\theta$, where θ is the angle between ∇T and $d\vec{r}$.



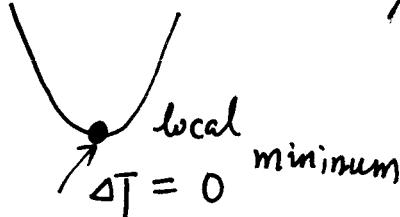
* Climb a hill. The direction of ∇h $h(x, y, z)$ is the steepest direction.

The magnitude of ∇h is the slope along this hardest direction.

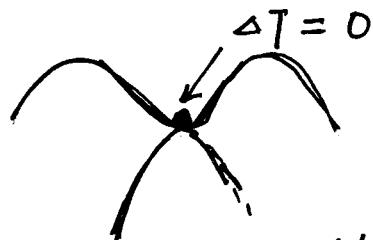
• Stationary point: $\nabla T = 0$. $dT \propto O(\Delta r)^2$



local maximum



local minimum



Saddle point
maximum along one direct
minimum along another dire

$$\text{Ex: } r = \sqrt{x^2 + y^2 + z^2}$$

$$\nabla r = \frac{\partial r}{\partial x} \hat{x} + \frac{\partial r}{\partial y} \hat{y} + \frac{\partial r}{\partial z} \hat{z} = \frac{1}{\sqrt{x^2 + y^2 + z^2}} (x \hat{x} + y \hat{y} + z \hat{z})$$

$$= \frac{\vec{r}}{r} = \hat{r}$$

→ nabla operator: $\nabla = \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z}$ (vector operator)

The effect of ∇ is: for any scalar function $f(x, y, z)$

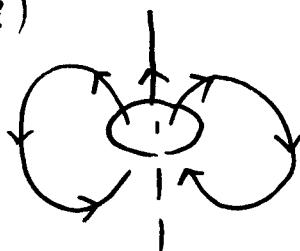
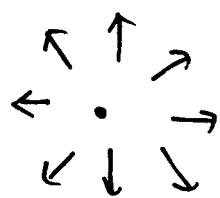
$$\nabla f = \hat{x} \frac{\partial f}{\partial x} + \hat{y} \frac{\partial f}{\partial y} + \hat{z} \frac{\partial f}{\partial z}$$

& divergence: — applying the ∇ -operator on a vector function: the point product

for a vector field $\vec{v} = v_x(x, y, z) \hat{x}$ each component
 $+ v_y(x, y, z) \hat{y}$ is a function
 $+ v_z(x, y, z) \hat{z}$ of (x, y, z)

Example: ① electric field $\vec{E}(x, y, z) = [E_x(x, y, z), E_y(x, y, z), E_z(x, y, z)]$

$$\textcircled{2} \quad \vec{B}(x, y, z)$$

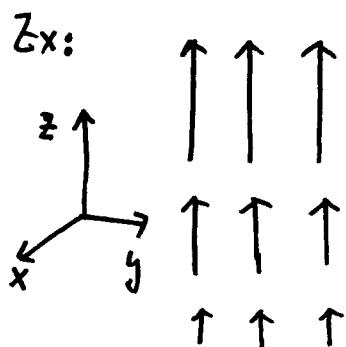


divergence $\nabla \cdot \vec{v} = (\hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z}) (v_x \hat{x} + v_y \hat{y} + v_z \hat{z})$

 $= \frac{\partial}{\partial x} v_x + \frac{\partial}{\partial y} v_y + \frac{\partial}{\partial z} v_z$

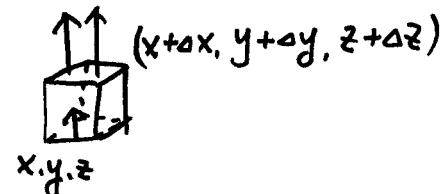
* a scalar function has no divergence.

* Physical meaning: $\nabla \cdot \vec{v}$ is a measure of the vector \vec{v} spreads out from a point ~~or~~.



$\vec{v} = (0, 0, v_z(z))$

$\nabla \cdot \vec{v} = \frac{\partial}{\partial z} v_z > 0.$



Consider a small box:

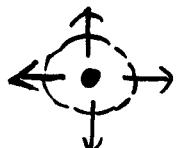
The flux of \vec{v} to outside
net

 $v_z(z+\Delta z) \Delta y \Delta x - v_z(z) \Delta y \Delta x$

$= \frac{\partial}{\partial z} v_z \Delta x \Delta y \Delta z$

$= \nabla \cdot \vec{v} (\Delta x \Delta y \Delta z)$

or Ex



$\vec{v} = (x, y, z) = \vec{r}$

Consider a small sphere with Δr

the flux goes outside $4\pi(\epsilon r)^2 \cdot \Delta r = 4\pi (\Delta r)^3$

$= \frac{4}{3}\pi (\epsilon r)^3 \cdot (\nabla \cdot \vec{r})$

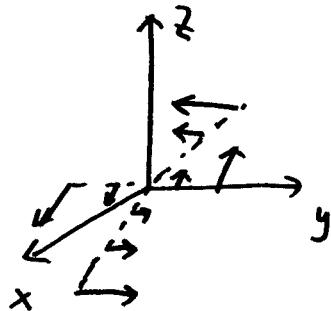
$= \Delta V (\nabla \cdot \vec{v})$

§3. Circulation

$$\nabla \times \vec{V} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix} = \hat{x} \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} \right) + \hat{y} \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x} \right) + \hat{z} \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right)$$

$\therefore \nabla \times \vec{V}$ is a measure of how much \vec{V} "curls around" the point.

The direction of $\nabla \times \vec{V}$ is perpendicular to the circulation plane

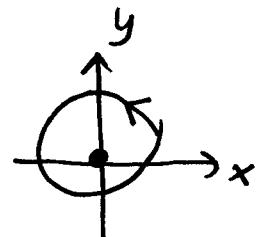


$$\text{Ex: } \vec{V} = (-y, x, 0)$$

Circulation around a circle

of radial Δr , along the ~~outward~~ counter-clockwise

$$\Delta r \cdot 2\pi \Delta r = \pi(\Delta r)^2 \cdot 2$$



counter-clockwise direction

$$\nabla \times \vec{V} = \hat{z} \cdot 2$$

$$\Rightarrow \text{circulation} = \Delta S \hat{z} \cdot (\nabla \times \vec{V})$$

④ Product rules

$$\left\{ \begin{array}{l} \nabla(f+g) = \nabla f + \nabla g \\ \nabla \cdot (\vec{A} + \vec{B}) = \nabla \cdot \vec{A} + \nabla \cdot \vec{B} \\ \nabla \times (\vec{A} + \vec{B}) = \nabla \times \vec{A} + \nabla \times \vec{B} \end{array} \right.$$

$$\begin{aligned} \nabla(kf) &= k \nabla f \\ \nabla \cdot (k\vec{A}) &= k \nabla \cdot \vec{A} \\ \nabla \times (k\vec{A}) &= k \nabla \times \vec{A} \end{aligned}$$

k is a const

linear operator
properties of

more $\nabla(fg) = f \nabla g + \nabla f g$

$$\nabla(f/g) = \frac{1}{g} \nabla f - \frac{f \nabla g}{g^2}$$

$$\nabla \cdot (f \vec{A}) = f(\nabla \cdot \vec{A}) + \vec{A} \cdot \nabla f$$

$$\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})$$

Note
required
to
remember

$$\nabla \times (f \vec{A}) = f(\nabla \times \vec{A}) - \vec{A} \times (\nabla f)$$

$$\nabla \times (\vec{A} \times \vec{B}) = (\vec{B} \cdot \nabla) \vec{A} - (\vec{A} \cdot \nabla) \vec{B} + \vec{A} (\nabla \cdot \vec{B}) - \vec{B} (\nabla \cdot \vec{A})$$

\therefore Laplacian : divergence of ~~the~~ gradient

$$\begin{aligned}\nabla \cdot (\nabla T) &= \frac{\partial}{\partial x} (\nabla T)_x + \frac{\partial}{\partial y} (\nabla T)_y + \frac{\partial}{\partial z} (\nabla T)_z \\ &= \frac{\partial^2}{\partial x^2} T + \frac{\partial^2}{\partial y^2} T + \frac{\partial^2}{\partial z^2} T = \nabla^2 T\end{aligned}$$

- Curl of gradient is zero

$$\nabla \times (\nabla T) = 0 \quad \rightarrow$$

$$[\nabla \times (\nabla T)]_z = \frac{\partial}{\partial x} (\nabla T)_y - \frac{\partial}{\partial y} (\nabla T)_x = \left(\frac{\partial^2}{\partial x \partial y} - \frac{\partial^2}{\partial y \partial x} \right) T = 0$$

similarly $[\nabla \times (\nabla T)]_x = [\nabla \times (\nabla T)]_y = 0$.

- Divergence of a curl is zero

$$\nabla \cdot (\nabla \times \vec{v}) = 0$$

$$\begin{aligned}\nabla \cdot (\nabla \times \vec{v}) &= \frac{\partial}{\partial x} [\nabla \times \vec{v}]_x + \frac{\partial}{\partial y} [\nabla \times \vec{v}]_y + \frac{\partial}{\partial z} [\nabla \times \vec{v}]_z \\ &= \frac{\partial}{\partial x} \left[\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right] + \frac{\partial}{\partial y} \left[\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right] + \frac{\partial}{\partial z} \left[\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right] \\ &= \left(\frac{\partial^2}{\partial x \partial y} - \frac{\partial^2}{\partial y \partial x} \right) v_z + \left(\frac{\partial^2}{\partial z \partial x} - \frac{\partial^2}{\partial x \partial z} \right) v_y + \left(\frac{\partial^2}{\partial y \partial z} - \frac{\partial^2}{\partial z \partial y} \right) v_x \\ &= 0\end{aligned}$$

(7)

- Curl of Curl

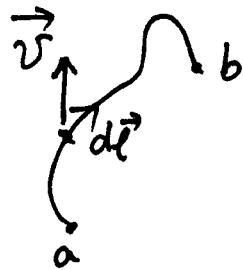
$$\nabla \times (\nabla \times \vec{U}) = \nabla(\nabla \cdot \vec{U}) - \nabla^2 \vec{U} \leftarrow \begin{matrix} \text{leave as} \\ \text{an exercise} \end{matrix}$$

definition of $\nabla^2 U^2 = (\nabla^2 U_x, \nabla^2 U_y, \nabla^2 U_z)$

Lect 3 Integral Calculus

§ Line integrals — path

$$\int_a^b \vec{v} \cdot d\vec{l}$$



if the path forms a closed loop $\rightarrow \oint_P \vec{v} \cdot d\vec{l}$.

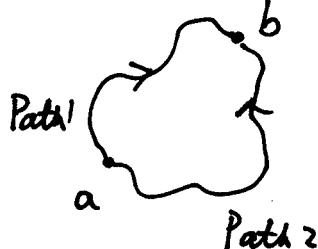
Example: work done by following a path $W = \int_a^b \vec{F} \cdot d\vec{l}$

Generally speaking, the integral $\int_a^b \vec{v} \cdot d\vec{l}$ depends on the path.

For a special class of vector field, such integrals are path-independent,
which means for a closed loop,

$$\int_a^b \vec{v} \cdot d\vec{l} = \int_a^b \vec{v} \cdot d\vec{l} \quad \text{the integral} = 0.$$

$$\Rightarrow \oint_{P_1 - P_2} \vec{v} \cdot d\vec{l} = 0$$

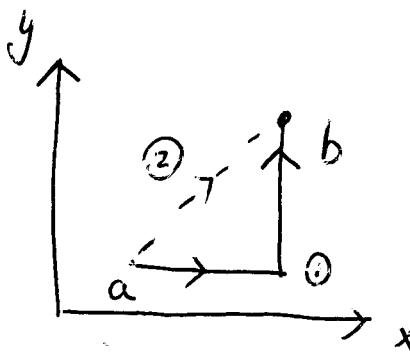


For force field satisfying this properties, we call it conservative force.

The static gravity, electrostatic force, .. etc are force!

Example: $\vec{v} = y^2 \hat{x} + 2x(y+1) \hat{y}$,

a(1,1), b(2,2)



following path 1

$$\int \vec{v} \cdot d\vec{l} = \int v_x dx + \int v_y dy = \int_1^2 dx + \int_1^2 4(y+1) dy$$

$(1,1) \rightarrow (2,1)$ $(2,1) \rightarrow (2,2)$

$$= 1 + (2y^2 + 4y) \Big|_1^2 = 1 + (16 - 6) = 11$$

following path 2: $\int \vec{v} \cdot d\vec{l} = \int_1^2 v_x dx + \int_1^2 v_y dy$

$$\rightarrow = \int_1^2 x^2 dx + \int_1^2 2y(y+1) dy = \frac{x^3}{3} \Big|_1^2 + \frac{2y^3}{3} + y^2 \Big|_1^2$$

plug in $x=y$

$$= \frac{8-1}{3} + \frac{8-1}{3} \times 2 + 4-1 = 7+3=10$$

if following ① and reverse ② and come back to a $\Rightarrow \oint \vec{v} \cdot d\vec{l} = 1$

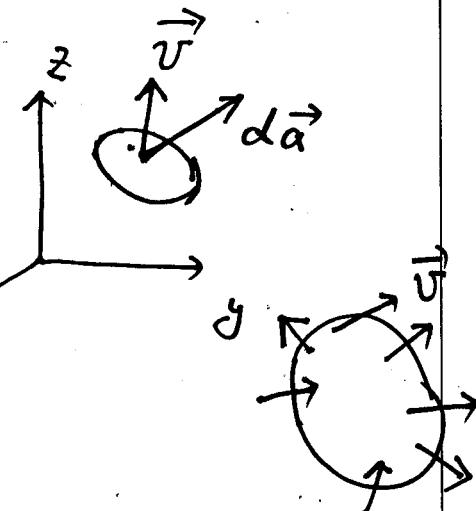
§ 2. surface integrals

$\int_S \vec{v} \cdot d\vec{a}$, $d\vec{a}$ is an infinitesimal area, the direction is along the normal x direction.

for a close surface

$\oint \vec{v} \cdot d\vec{a}$, the normal direction: from inside to outside.

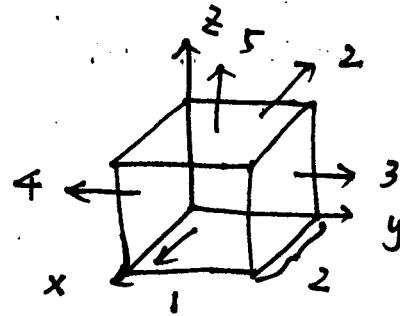
If \vec{v} describe a flow of a liquid, then $\oint \vec{v} \cdot d\vec{a}$ is the flux.



$$\text{Ex 1.7} \quad \vec{v} = 2xz \hat{x} + (x+2) \hat{y} + y(z^2 - 3) \hat{z}$$

a cube with edge length 2.

the 5-surfaces (except the bottom)
form a big surface, calculate $\int \vec{v} \cdot d\vec{a}$



$$\int \vec{v} \cdot d\vec{a} = \int_1 + \int_2 + \dots + \int_5$$

$$\int_1 \vec{v} \cdot d\vec{a} = \int v_x dy dz = \int 4z dy dz = \int_0^2 dy \int_0^2 4z dz = 2 \cdot 2z^2 \Big|_0^2 = 16$$

\uparrow set $x=2$

$$\int_2 \vec{v} \cdot d\vec{a} = - \int v_x dy dz \Big|_{set x=0} = 0$$

$$\int_3 \vec{v} \cdot d\vec{a} = \int v_y dx dz \Big|_{set y=2} = \int_0^2 (x+2) dx \int_0^2 dz = \left(\frac{x^2}{2} + 2x \right) \Big|_0^2 \cdot 2 = 12$$

$$\int_4 \vec{v} \cdot d\vec{a} = - \int v_y dx dz \Big|_{set y=2} = - \int_0^2 (x+2) dx \int_0^2 dz = -12$$

$$\int_5 \vec{v} \cdot d\vec{a} = \int v_z dx dy \Big|_{set z=2} = \int_0^2 \int_0^2 y dx dy = 2 \cdot \frac{y^2}{2} \Big|_0^2 = 4$$

$$\Rightarrow \int_1 + \dots + \int_5 = 16 + 4 = 20$$

Volume integrals $\int_V T dz$ where $dz = dx dy dz$
 T is a scalar function

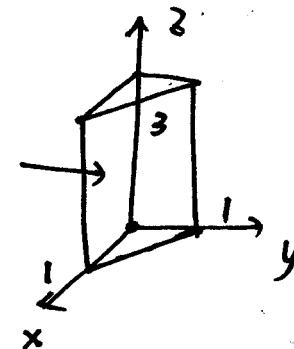
Sometimes we also calculate volume integral of vectors

$$\int \vec{v} dz = (\int v_x dz) \hat{x} + (\int v_y dz) \hat{y} + (\int v_z dz) \hat{z}$$

Ex 1.8 :

$$T = xyz^2, \quad \int_V T dz \text{ for the prism}$$

$$\int_0^3 dz \int_0^{1-x} dy \int_0^1 dx \cdot xyz^2$$



$$= \int_0^3 z^2 dz \int_0^1 x dx \int_0^{1-x} y dy = \frac{z^3}{3} \Big|_0^3 \cdot \int_0^1 dx \cdot x \frac{(1-x)^2}{2}$$

$$= \frac{9}{2} \cdot \int_0^1 (x - 2x^2 + x^3) dx = \frac{9}{2} \cdot \left[\frac{1}{2} - \frac{2}{3} + \frac{1}{4} \right] = \frac{9}{2} \cdot \frac{1}{12} = \frac{3}{8}$$