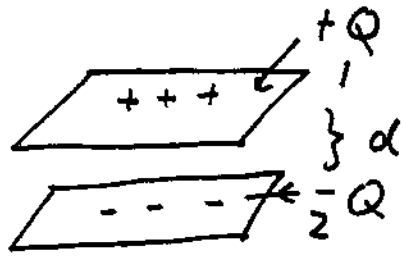
Lect 17 Dielectrics

Consider a capacitor $C = \frac{Q}{V_{12}} = \frac{A}{4\pi d}$.



if we insert insulating material, we will find V_{12} is weakened, and thus C increases. The difference can be summerize by a constant — ε (dielectric constant).

$$E_{in\text{-}media} = \frac{E_{vaccum}}{\varepsilon} \Rightarrow C_{with\text{-}media} = \varepsilon C_{vaccum}.$$

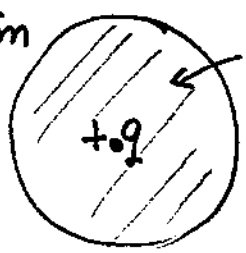
dielectric constant for comman materials

H_2O	80,	S	4.0
Benine	2.28	NH_3	22
NaCl	6.12.	rubber	~4
		wax	2.1 ~ 2.5

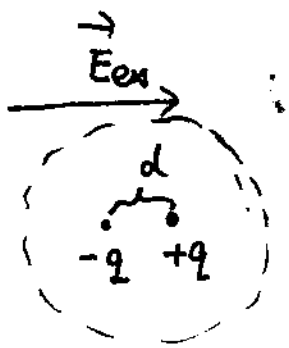
we need to ~~construct~~ a theory to describle the effect of E-field in the medium. In the dielectric material, there's no free charge, thus the external electric field cannot be perfectly screened as in metal. Nevertheless, electric-polarization will be induced which partially cancel the external field.

§ dipole of atom & and molecules

atom



$-q$, uniformally distribute around nucleus.



assum the atom raidus a, and charge uniformally distributed

$\Rightarrow$ now nucleus at a distance "d" from the center

$\Rightarrow$ E at distance d is $\frac{q^2}{a^2} \cdot \frac{d}{a}$, which should be equal to E_{ex}

$\Rightarrow$ $\frac{qd}{a^3} = E_{ex}$ $\Rightarrow$ dipole $p = qd = a^3 E_{ex} = \frac{3}{4\pi} Vol \cdot E_{ex}$

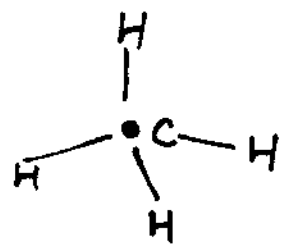
define polarizbility α $\Rightarrow$ $p = \alpha E$, where $\alpha \approx a^3$.

α carries the unit of volume. If we express $\frac{\alpha}{a_0^3}$ which a_0 is Bohr radius

$\Rightarrow$ $\frac{\alpha}{a_0^3}$ = 0.66 (H), 0.2 (He), 24.3 (Li), 43.4 (K)
59.6 (Cs), 1.76 (C), 24.1 (Na), 0.396 (Ne), 1.64 (Ar)

how about molecule:

molecule has a concrete shape: no-longer spherical for molecules with high symmetries, say CH_4, the induced dipole is parallel to external field



$$\vec{p} = \alpha \vec{E}_{ex}.$$

However, this is not always the case, say, CO_2.

The responses for the field along the bond $O=C=O$. or perpendicular to the bond is quite different! $\Rightarrow$

$$\vec{p} = \alpha_{\perp} \vec{E}_{\perp} + \alpha_{\parallel} \vec{E}_{\parallel}. \quad \text{thus } \vec{p} \nparallel \vec{E}.$$

generally $\begin{pmatrix} P_x \\ P_y \\ P_z \end{pmatrix} = \begin{pmatrix} \alpha_{xx} & \alpha_{xy} & \alpha_{xz} \\ \alpha_{yx} & \alpha_{yy} & \alpha_{yz} \\ \alpha_{zx} & \alpha_{zy} & \alpha_{zz} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$

↑ polarizibility tensor.

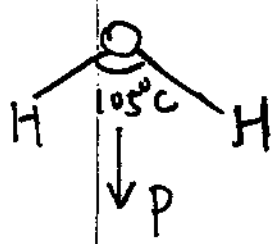
*Later on we can prove that α_{ij} is symmetric, i.e. $\alpha_{ij} = \alpha_{ji}$.

we can always find 3-priciple axes, along which $\vec{P} \parallel \vec{E}$, in other words. as coordinate in which α is diagonal $= \begin{pmatrix} \alpha_{x'x'} & & 0 \\ & \alpha_{y'y'} & \\ 0 & & \alpha_{z'z'} \end{pmatrix}$.
(x', y', z')

c.f. the moment of inertial, principle axes.

§ Pemanet electric dipole moment ↙ non-polar molecule

many molecules CO_2, CH_4 has no intrincic dipole moment, We need external electric field to induce a dipole ment. But for HCl, H_2O, their positive charge and negative charge centers do not coincide ⇒ polar-molecule



$P \approx 6.1 \times 10^{-30}$ C.m

other polar molecules

HCl	$P_{HCl}/P_{H_2O} \approx 0.56$
NH_3	$P/P_{H_2O} \approx 0.78$
CH_3OH	$P/P_{H_2O} \approx 0.92$

☆ but actually even for a H atom, at any instant time, it has an dipole moment. But electrons moves so fast that for usual purpose, we observe an average effect so that dipole moment averges to zero.

Rigorously speaking, if we wait for long enough time, the dipole moment of H_2O, HCl, etc should also averge to zero. However, this is the time scale related to the relative motion of atoms inside molecule, which is much slower than the time-scale of electron motion.

For the interaction between molecules, its chacteristic time scale is not long enough to allow to average out all the molecule configuration, thus we can think polar-molecules with a "permanent" dipole moment.

But for atom, we do think that it's diapole averges to zero.

For polar molecule, under the external field $\vec{E}$, if $\vec{P} \not\parallel \vec{E}$, it feels a torge.

$$\vec{N} = \vec{r}_+ \times \vec{F}_+ + \vec{r}_- \times \vec{F}_- = q(\vec{r}_+ - \vec{r}_-) \times \vec{E} = \vec{p} \times \vec{E}$$

total force $\vec{F} = \vec{F}_+ - \vec{F}_- = q(\vec{E}_+ - \vec{E}_-) = q \Delta \vec{E}$

$$= q\left(\hat{x}[\partial_x E_x \cdot d_x] + \hat{y}(\partial_y E_y \cdot d_y] + \hat{z}(\partial_z E_z \cdot d_z)\right)$$

$$= q(\vec{d} \cdot \vec{\nabla})\vec{E} = (\vec{p} \cdot \vec{\nabla})\vec{E}$$

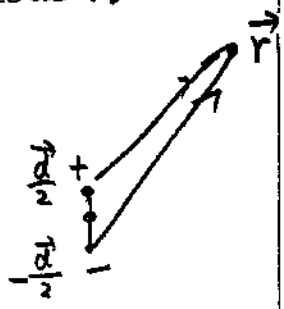
§ Polarization:

$$P = \text{dipole moment} / \text{unit volume}.$$

Lect 18 Field of a Polarized Object

Suppose that we know the distribution of $\vec{P}(\vec{x})$, how is the electric field /potential generated by $\vec{P}(\vec{x})$? we will study such a question.

For a single dipole $V(\vec{r}) = \dfrac{q}{|\vec{r}-\frac{\vec{d}}{2}|} - \dfrac{q}{|\vec{r}+\frac{\vec{d}}{2}|}$

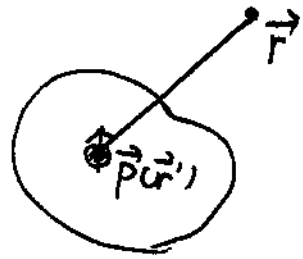


$$= q\,\nabla\frac{1}{r}\cdot(-\vec{d}) = -\vec{P}\cdot\nabla\frac{1}{r}$$

$$= \frac{\vec{P}\cdot\hat{r}}{r^2}$$

Then we apply to a distribution of $\vec{P}(\vec{r})$, by replacing $\vec{P}$ with $\vec{P}(r)d\tau$

$$\Rightarrow V(\vec{r}) = \int_V \frac{(\vec{r}-\vec{r}')\cdot\vec{P}(\vec{r}')}{|\vec{r}-\vec{r}'|^3}\,d\tau'$$



$$= \int \vec{P}(\vec{r}')\cdot\nabla'_{r'}\frac{1}{|\vec{r}-\vec{r}'|}\,d\tau'$$

$$= \int \frac{-\nabla_{r'}\cdot\vec{P}(\vec{r}')}{|\vec{r}-\vec{r}'|}\,d\tau' + \int \nabla'\left(\frac{\vec{P}(\vec{r}')}{|\vec{r}-\vec{r}'|}\right)d\tau'$$

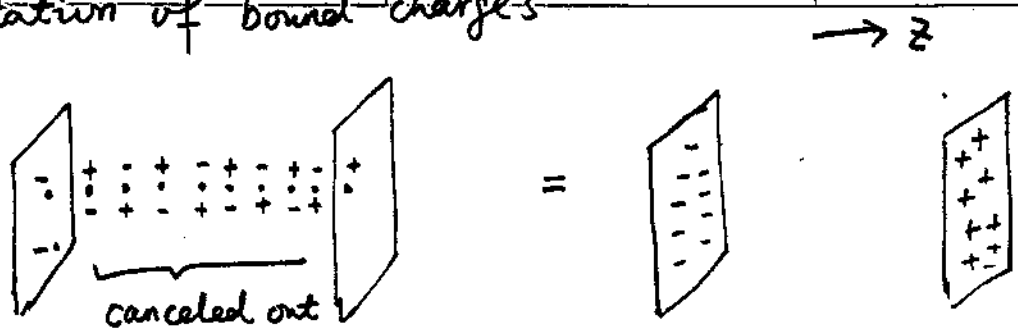
$$= \int d\tau' \frac{-\nabla_{r'}\cdot\vec{P}(\vec{r}')}{|\vec{r}-\vec{r}'|} + \oint_S \frac{\vec{P}(\vec{r}')\cdot d\vec{a}'}{|\vec{r}-\vec{r}'|}$$

we can integrate it as $V(\vec{r})$ is generated by a body charge density and surface charge density

bound charges

$$V(r) = \int d\tau' \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} + \oint_S \frac{\sigma(\vec{r}')\,da'}{|\vec{r}-\vec{r}'|}, \quad \text{where } \boxed{\begin{aligned}\rho(\vec{r}') &= -\nabla\cdot\vec{P}(r)\\ \sigma(\vec{r}') &= \vec{P}(r)\cdot\hat{n}\end{aligned}}$$

Interpretation of bound charges



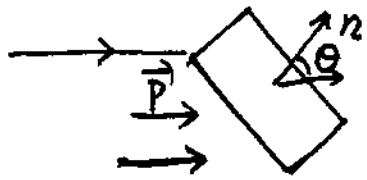
uniform $\vec{P}$

$\rho = \nabla\cdot\vec{P} = 0$

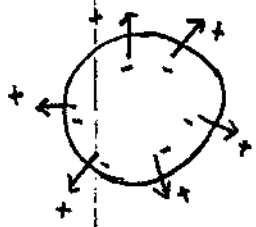
$\sigma = \vec{P}\cdot(-\hat{z}) = -P$

$\sigma = \vec{P}\cdot\hat{z} = P$

if for



$\Rightarrow \sigma_b = \vec{P}\cdot\hat{n} = P\cos\theta$



$Q = \int \rho_b \, d\tau = -\oint \vec{P}\cdot da \Rightarrow \rho_b = -\nabla\cdot\vec{P}$

Exercise: electric field distribution of a uniformly charged ball

Solution: $\rho_b = -\nabla\cdot\vec{P} = 0$

$\sigma_b = \vec{P}\cdot\hat{n} = P\cos\theta$.

As shown before, this can be mapped to the solution of Laplace equation $\nabla^2 V = 0$, subject to the boundary condition $\sigma(\theta) = P\cos\theta$ on the sphere. Only the $l=1$ component exist

as

$$V(r,\theta) = \begin{cases} \frac{4\pi P}{3} r\cos\theta & r \le R \\ \frac{4\pi P}{3}\frac{R^3}{r^2}\cos\theta & r \ge R. \end{cases}$$

check

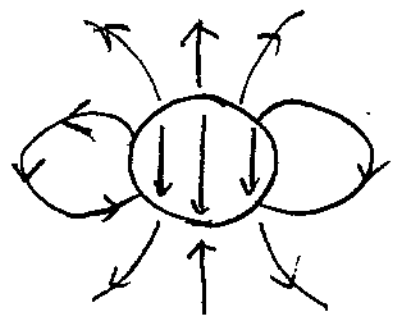
$$\Rightarrow \left.\frac{\partial V}{\partial r}\right|_{r=R^+} - \left.\frac{\partial V}{\partial r}\right|_{r=R^-} = \frac{4\pi P}{3} 3\cos\theta = 4\pi\sigma$$

$$\Rightarrow \sigma(\theta) = P\cos\theta.$$

The field inside the ball is uniform $\vec{E} = -\nabla V = -\frac{4\pi}{3}\vec{P}$.

The potential outside is a dipole-like $V = \frac{\vec{P}\cdot\hat{r}}{r^2}$ with $\vec{P} = \frac{4}{3}\pi R^3 \vec{P}$.

The field at $r > R$ $\quad \vec{E}_{dip}(r,\theta) = \frac{P}{r^3}(2\cos\theta\,\hat{e}_r + \sin\theta\,\hat{e}_\theta)$



* Justification of using $\left.\begin{array}{l}\rho_b = -\nabla\cdot\vec{P} \\ \sigma_b = \vec{P}\cdot\hat{n}\end{array}\right\}$

to calculate the field. Clearly it's fine to use (ρ_b, σ_b) to describe the effects of induced charge for points outside the material.

$\vec{P}$ can only be defined as an average over a micro-scopically large but macroscopically small area.

$$\vec{P}(\vec{r}) = \sum_{\text{for } r_i \text{ close to } \vec{r}} \vec{P}_i / \Delta V .$$

ΔV is large so that $\vec{P}(\vec{r})$ is smooth by average and is also small so that $\vec{P}(\vec{r})$ doesn't vary much. micro-scopic $\vec{P}_i$, which fluctuates strongly

For points outside the material, naturally it is fine to use the smooth ρ_b and σ_b to represent the effect of media.

However, for points inside the media, microscopically $\vec{E}_{mi}$ can be very large if $\vec{r}$ is close to electrons, but it fluctuats strongly. We are not interested in its detailed distribution. Instead, we use a coarse average definition $\vec{E}(\vec{r}) = \int_{\text{close to } \vec{r}} \vec{E}_{mi}(\vec{r}')d\vec{r}' / \Delta V$. this macroscopic field is what we are interested

This smoothrized fields $\vec{E}(\vec{r})$ are what really described by σ_b and ρ_b.

We will introduce a new quantity to describle electricity in the media.

* electric Displacement

$$\vec{D} = \vec{E} + 4\pi\vec{P}$$

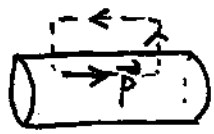
$$\nabla\cdot\vec{D} = \nabla\cdot\vec{E} + 4\pi\underset{(\nabla\cdot}{\vec{P}}) = 4\pi(\rho_f+\rho_b) - 4\pi(\rho_b) = 4\pi\rho_f$$

so $\vec{D}$ is determined purely from ρ_f.

The Gauss's law for $\vec{D}$: $\oint \vec{D}\cdot d\vec{a} = 4\pi Q_f$.

however, even for electro-statics, $\nabla\times\vec{D} = 4\pi\,\nabla\times\vec{P} \neq 0$ in general.

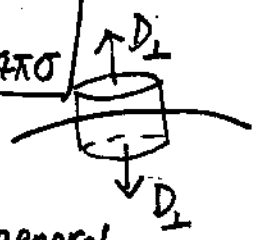
there's no potential for $\vec{D}$.



$\oint \vec{P}\cdot d\vec{l} \neq 0$

boundy conditions for $\vec{D}$. c.f. $E_{\perp,\text{above}} - E_{\perp,\text{below}} = 4\pi\sigma$

$$\boxed{D_{\perp,\text{above}} - D_{\perp,\text{below}} = 4\pi\sigma_f .}$$



The // component of D dues n't vanish in general

$$D''_{\text{above}} - D''_{\text{below}} = E_{\parallel\,\text{above}} - E_{\parallel\,\text{below}} + 4\pi(\vec{P}_{\text{above}\parallel} - \vec{P}_{\text{below}\parallel})$$

$$= 4\pi(\vec{P}_{\text{above}\parallel} - \vec{P}_{\text{below}\parallel}),$$ but this no useful.

we use $\boxed{E_{\parallel,\text{above}} = E_{\parallel,\text{below}}}$

examp: Prob 4.15 a thick spherical shell with inner radius a, outer radius b is made of dielectric material with a "frozen-in" polarization

$$p(r) = \frac{k}{r}\hat{r}, \quad \text{where } k \text{ is a constant.}$$

we find the charge distribution and electric field distributions in the 3 regions.

1° we compute the bound charge density directly

$$\rho_b(r) = -\nabla\cdot p(r) = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{k}{r}\right) = -\frac{k}{r^2}$$

$$\sigma_b(r=a) = -\vec{p}\cdot\hat{n} = -\frac{k}{a}$$

$$\sigma_b(r=b) = +\vec{p}\cdot\hat{n} = \frac{k}{b}$$

check the total bound charge $4\pi\left[\int_a^b \rho_b(r)\, r^2 dr - \frac{k}{a}\cdot a^2 + \frac{k}{b}\cdot b^2\right] = 0.$

we can apply Gauss's theorem, draw a sphere around the center and calculate the electric flux

1° for $r<a$. $\oint \vec{E}\cdot da = 0 \Rightarrow E(r) = 0$

2° for $a<r<b$ $\oint \vec{E}_{(r)}\cdot da = \sigma_a\cdot 4\pi a^2 + \int_a^r \rho(r)\, r^2 dr\cdot 4\pi$

$$= (4\pi)\left[-k a - k\int_a^r dr\right] = -4\pi r \Rightarrow \vec{E}(r) = -\frac{4\pi k}{r}\times\hat{r}$$

3° for $r>b$ $\frac{1}{4\pi}\oint \vec{E}\cdot da = \sigma_a\cdot 4\pi a^2 + \int_a^r \rho(r)\, r^2 dr\cdot 4\pi + \sigma_b\, 4\pi b^2 = 0$

$$\Rightarrow \vec{E}(r) = 0$$

2° we can get the same electric field distribution by use D. Since there're no free charge, $\oint \vec{D}\cdot d\vec{a} = 0$. Due to the spherical

Symmetry, $\vec{D}$ is along the radial direction. $\Rightarrow \vec{D}(r) = 0$ everywhere.

at $r < a$ or $r > b$, it's vacum $\Rightarrow \vec{E}(r) = \vec{D}(r) = 0$

at $a < r < b$ $\Rightarrow \vec{D} = \vec{E} + 4\pi\vec{P} = 0 \Rightarrow \vec{E} = -4\pi\vec{P} = -4\pi\frac{k}{r}\hat{r}$.