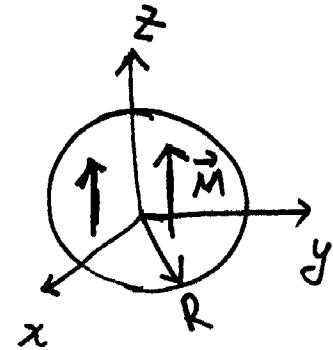


Lecture 9: Examples on magnetic fields

example 1: find the magnetic field of a uniformly magnetized sphere.

set \vec{M} along the \hat{z} -axis. $\Rightarrow \vec{j}_b = c \nabla \times \vec{M} = 0$

$$\vec{k}_b = c \vec{M} \times \hat{n} = c M \sin\theta \hat{e}_\phi$$



This current distribution is the same as the

uniformly charged sphere under rotation $\vec{k} = \sigma v = \sigma \omega R \sin\theta \hat{e}_\phi$

we identify $cM = \sigma \omega R$, we have inside the sphere

$$\vec{B}_{\text{inside}} = \frac{8\pi}{3c} R \sigma \omega \hat{z} = \frac{8\pi}{3} \frac{R \sigma \omega}{c} \hat{z} = \frac{8\pi}{3} \vec{M}$$

$$\begin{aligned} \vec{B}_{\text{outside}} &= \frac{3(\vec{M}_{\text{tot}} \cdot \hat{r}) \hat{r} - \vec{M}_{\text{tot}}}{r^3} \quad \text{with} \quad \vec{M}_{\text{tot}} = \frac{4\pi}{3c} R^4 \sigma \omega \hat{z} \\ &= \frac{4\pi}{3} R^3 \frac{\sigma \omega R}{c} \hat{z} \\ &= \frac{4\pi}{3} R^3 M \hat{z} \end{aligned}$$

We can also solve this problem by using magnetic potential, W.
scalar

$\nabla \times \vec{H} = 0$ in our problem. We can write

$$\vec{H} = -\nabla W. \quad \nabla \cdot \vec{H} = \nabla \cdot (\vec{B} - 4\pi \vec{M}) = -4\pi \nabla \cdot \vec{M}$$

$$\Rightarrow -\nabla^2 W = -4\pi \nabla \cdot \vec{M} \Rightarrow \text{Poisson equation.}$$

(2)

$\nabla \cdot \vec{M} = 0$ everywhere, except on the boundary. We need the following boundary conditions

$$\textcircled{1} \quad W_{\text{in}}(R, \theta) = W_{\text{out}}(R, \theta) \quad \leftarrow \vec{w}(b) - \vec{w}(a) = \int_a^b \nabla w d\ell = - \int_a^b H d\ell$$

$$\textcircled{2} \quad \rightarrow 0 \text{ as } b \rightarrow a$$

$$\Rightarrow - \frac{\partial W_{\text{out}}}{\partial r} \Big|_R + \frac{\partial W_{\text{in}}}{\partial r} \Big|_R = 4\pi \sigma_M = 4\pi \vec{M} \cdot \hat{e}_r = 4\pi M \omega s \theta$$

in and outside the sphere, we solve Laplace equation $\nabla^2 W = 0$ using the method of variable separation.

$$W(r, \theta, \phi) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) P_\ell(\cos \theta)$$

at $r < R$, we take the branch of r^ℓ , $w(r, \theta, \phi) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos \theta)$

$r > R$ we take the $r^{\ell+1}$, $w(r, \theta, \phi) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos \theta)$

$$\text{at } r=R \Rightarrow A_\ell R^\ell = \frac{B_\ell}{R^{\ell+1}} \Rightarrow B_\ell = R^{\ell+1} A_\ell$$

$$\sum (\ell+1) \frac{B_\ell}{R^{\ell+2}} P_\ell(\cos \theta) + \sum \ell A_\ell R^{\ell-1} P_\ell(\cos \theta) = 4\pi M \omega s \theta$$

$$\Rightarrow \left\{ \begin{array}{l} \frac{2B_1}{R^3} + A_1 = 4\pi M, \quad \text{and all other } A_\ell = B_\ell = 0 \\ B_1 = A_1 R^3 \end{array} \right. \Rightarrow A_1 = \frac{M}{3} 4\pi$$

$$\Rightarrow W_{in}(r, \theta, \varphi) = \frac{4\pi M}{3} r \cos\theta = \frac{4\pi}{3} M z$$

$$\vec{H} = -\nabla W_{in} = -\frac{4\pi}{3} M \hat{z}$$

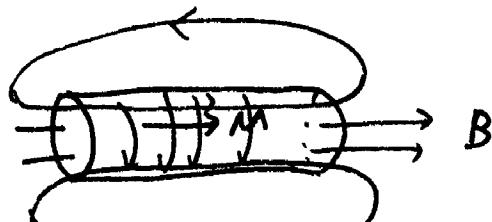
$$\vec{B} = \vec{H} + 4\pi \vec{M} = \left(\frac{4\pi}{3} + 4\pi\right) M = \frac{8\pi}{3} M \hat{z}$$

$$W_{out}(r, \theta, \varphi) = \frac{B_i}{r^2} P_i(\cos\theta) = \frac{\omega_0 \theta}{r^2} \frac{4\pi M R^3}{3} = \frac{z}{r^3} \cdot M_{tot}$$

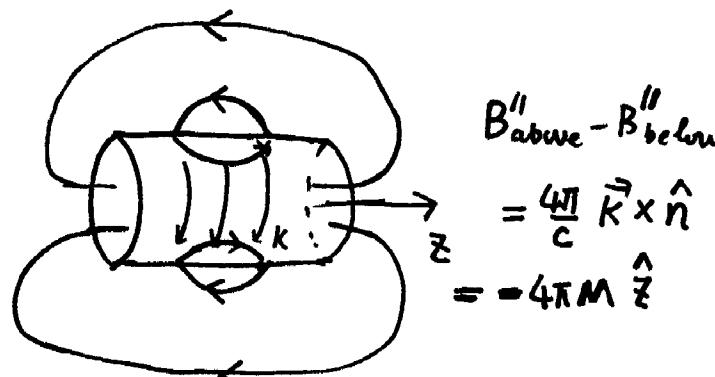
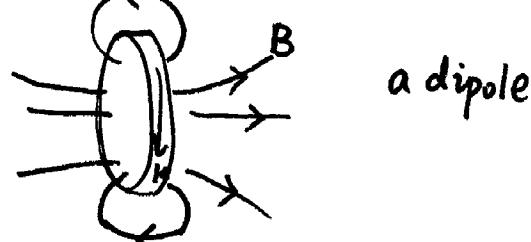
$$\Rightarrow \vec{H} = \vec{B} = -\nabla W_{out} = \frac{3(\vec{M}_{tot} \cdot \hat{r}) \hat{r} - \vec{M}_{tot}}{r^3}$$

example 2: Problem 6.9 a short circular of radius a and length L carries a "frozen-in" uniform magnetization \vec{M} parallel to its axis. Find the bound currents, and sketch the magnetic field of the cylinder (for $L \gg a$, $L \ll a$ and $L \approx a$).

$$\vec{j}_b = c \nabla \times \vec{M} = 0, \quad \vec{K}_b = c \vec{M} \times \hat{n} = c M \hat{e}_\phi$$



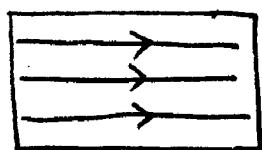
(a long solenoid $L \gg a$)



(4)

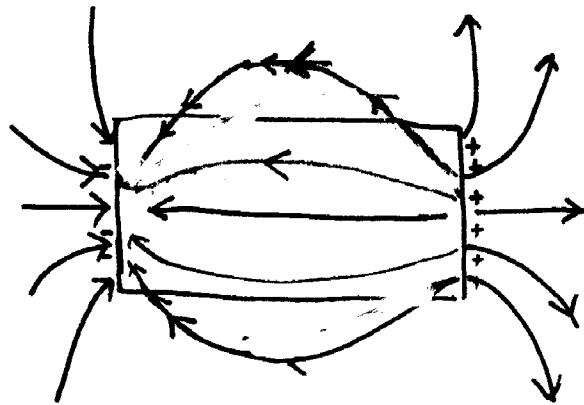
for $L \approx 2a$.

$$\mathbf{B} = \mathbf{H} + 4\pi \mathbf{M}$$

 M 

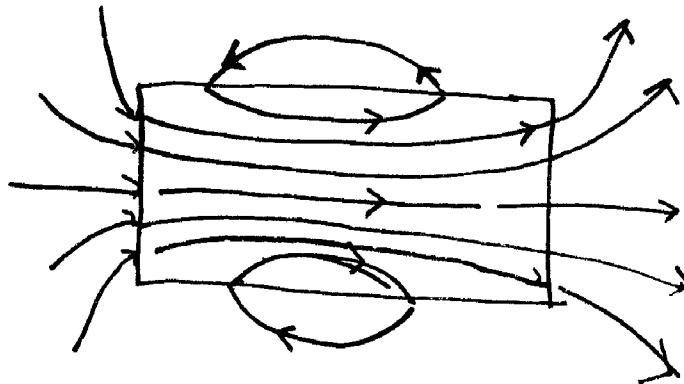
$$\nabla \cdot \vec{H} = - \underbrace{4\pi \nabla \cdot \mathbf{M}}_{P_m}$$

$\Rightarrow \vec{H}$ is field lines of uniform polarization charge



$$H''_{\text{above}} - H''_{\text{below}} = 0$$

$$H^\perp_{\text{above}} - H^\perp_{\text{below}} = -(M^\perp_{\text{above}} - M^\perp_{\text{below}})$$



$$\mathbf{B} = \mathbf{H} + 4\pi \mathbf{M}$$