

Lect 2. electrostatics — charges

- Charges (electric & magnetic)
- Charge conservation
- Charge quantization
- Charge invariance
- electrostatic energy, (potential energy)
- Madelung constant

§1 Electric charge

- The existence of positive and negative charges

c.f. gravity (only attraction), chromodynamics (R, G, B)

particle \rightarrow anti-particle, charge flips its sign. "C"

CPT: charge conjugation / parity / time-reversal

- so far, magnetic monopole has not been discovered yet!

but there's no fundamental principle forbidding its existence.

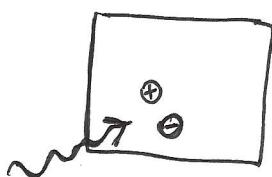
- under P (parity), monopole charge $\begin{cases} g \rightarrow -g \\ e \rightarrow e \end{cases}$

T (time-reversal) $\begin{cases} g \rightarrow -g \\ e \rightarrow e \end{cases}$

- Charge conservation, quantization, invariance.

§2. Charge conservation

charge cannot be created, and cannot be annihilated. If a charge is observed to appear at a certain place, it must come from a place nearby, or, an opposite charge moves to other place.



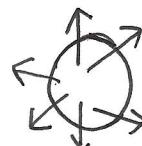
a γ -ray can excite an electron - positron pair from the vacuum.

a light can excite an electron - hole pair in semiconductor.

→ photo galvanic effect, solar cell, photosynthesis

- Charge conservation is a local effect!

$$\frac{\partial Q}{\partial t} + \oint d\vec{s} \cdot \vec{j} = 0$$



Consider a volume V surrounded by a surface C .

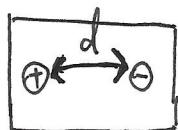
Then the charge change within the volume C equals the sum of charge flowing in- and outside

This result applies to an arbitrarily small volume.

For example, if consider such a phenomenon, that a pair of charges observed created at the same time but at a distance. This can not happen! They must be created at the same place, and then separate.



$t < t_0$



$t = t_0$

(This cannot happen!

Actually, it also absurd from relativistic point of view.)

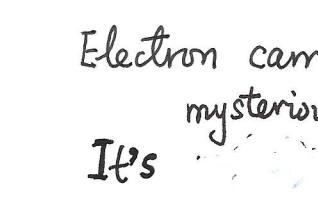
§b. Charge invariance

Electric charge is a relativistic invariant quantity. This is far from obvious, in contrast, mass is not invariant under relativistic transformation. Consider a hydrogen molecule, the motion of electron and proton differ by a factor of the mass ratio.  Yet, the hydrogen atom remains charge neutral.

§c. Charge quantization

① electric charge is quantized in the unit of e .

Electron carries charge $-e$, and proton carries charge $+e$.
mysterious

It's *mysterious* that why the proton charge and electron charge are exactly opposite? If they differ by a small fraction, at the macroscopic scale, matter is not neutral. 

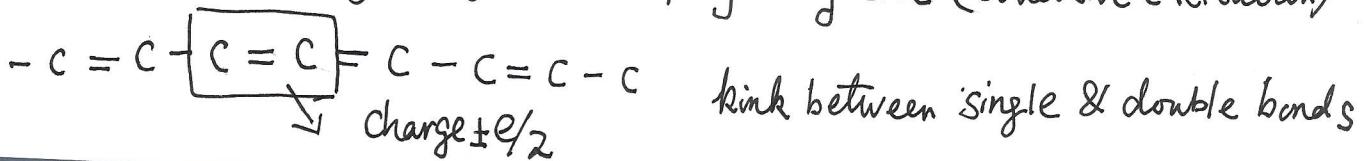
② Quarks carry fractional charge $\pm \frac{1}{3}e$, or $\pm \frac{2}{3}e$.

proton



$u:$	$\frac{2}{3}e$
$d:$	$-\frac{1}{3}e$

③ 1d conducting polymer - polyacetylene (collective excitation)



S₂ Coulomb's law



By symmetry principle and the uniqueness of electrostatics, there's only a unique force by Q at q . The force should be invariant by rotation around the " $Q-q$ " line, otherwise the force won't be unique.

$$\vec{F} = \frac{1}{4\pi\epsilon_0} \frac{Qq}{r^2} \hat{r} \quad - \text{SI unit}$$

$$\left\{ \begin{array}{l} \frac{Qq}{r^2} \hat{r} \\ \end{array} \right. \quad - \text{Gauss unit}$$

electric field: a test charge \checkmark feels a force due to other electric charges and their motions. If the test charge is fixed, (velocity = 0), then we define $\vec{E} = \vec{F}/q$.

Actually, electrons have spin which generates magnetic moment. Hence, the interaction between two fixed electrons has the magnetic part in addition to the electrostatic part. Typically, the magnetic dipolar interaction is much smaller. (how weak?)

If the test charge is moving, it experiences another type of force due to the magnetic field — the Lorentz force

$$\vec{F} = q \vec{v} \times \vec{B} \quad \text{or} \quad \vec{F} = q \frac{\vec{v}}{c} \times \vec{B}.$$

④ Superposition law — electric charges are additive



$$\vec{F}_3 = \frac{q_1 q_3}{r_{31}} \hat{r}_{31} + \frac{q_2 q_3}{r_{32}} \hat{r}_{32} \rightarrow \frac{q_1 (q_2 + q_3)}{r_{33}} \hat{r}_{33}$$

- energy of many charges

why?

For electrostatic system, the electric force is conservative. Suppose two charges q_1 and q_2 are separated by a distance r_{12} . The potential energy equals the work done to bring two charges from infinity to the distance r .

to overcome the electrostatic force

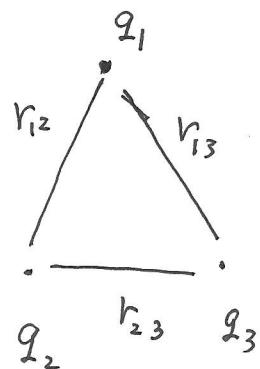
$$U_1 = - \int_{\infty}^{r_{12}} \vec{F}_{12} \cdot d\vec{r} = \frac{q_1 \cdot q_2}{r_{12}}. \quad (\text{Gauss unit})$$

Then bring the 3rd charge q_3 from infinity to a finite distances to q_1, q_2 , the work done is

$$U_2 = - \int_{\infty}^{r_2} \vec{F}_{23} \cdot d\vec{r} - \int_{\infty}^{r_3} \vec{F}_{13} \cdot d\vec{r} = \frac{q_3(q_1)}{r_{13}} + \frac{q_3 q_2}{r_{23}}$$

Hence, the total potential energy

$$U = \frac{q_1 q_2}{r_{12}} + \frac{q_2 q_3}{r_{23}} + \frac{q_3 q_1}{r_{31}}$$



① We assume that the potential energy is zero

when all charges are separated at infinite distances.

② U is symmetric with respect to 1, 2, 3. Hence, it is independent on how we bring charges together.

If we have n -charges, the energy is $U = \frac{1}{2} \sum_{i \neq j} \frac{q_i q_j}{r_{ij}}$.

★ Potential (electrostatic) of crystal — Madelung constant

For simplicity, let us consider a one dimensional crystal.

$$\begin{aligned} \frac{U}{N} &= \frac{1}{2} e^2 \left[-\frac{1}{a} + \frac{1}{2a} - \frac{1}{3a} + \dots \right] \times 2 \\ &= -\frac{e^2}{a} \left[1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \dots \right] = -\frac{e^2}{a} \ln 2 \end{aligned}$$

The electrostatic energy is negative, which partly explain the binding energy of material. But electrostatic energy cannot be the whole story, since it does not have a minimum! If let $a \rightarrow 0$, the electrostatic energy would go to negative infinity! We need quantum theory to explain the stability of matter, which is due to Pauli's exclusion principle of fermions. (Hard to believe, isn't it?) Later, we will prove that by electrostatics itself, a system consisting of positive and negative charges cannot reach equilibrium.

HW Consider a 2D crystal in a square lattice, calculate its Madelung constant

