

Lect 3 - Gauss's law

- Electric flux

① Static charge : For a sphere with the radius R ,
the flux $\Phi = \frac{q}{r^2} \cdot 4\pi r^2 = 4\pi q$.

If the surface is not a sphere, we draw a small sphere with radius r_0 , inside the surface. Consider an element of solid angle $d\Omega$.

It cut the small sphere at area of $d\vec{a}$, and the surface at the area $d\vec{A}$.

The flux penetrating the area of $d\vec{a}$

$$d\Phi = \frac{q}{r_0^2} da = q d\Omega$$

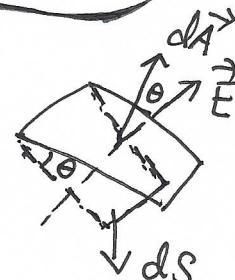
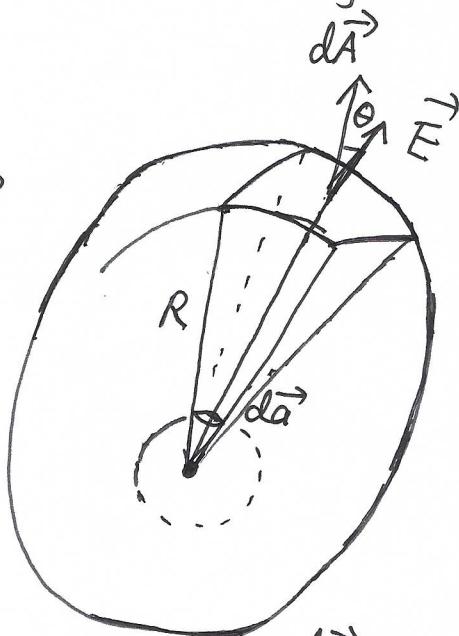
As for the area $d\vec{A}$, it's
projection to the direction of \vec{E}

$$\text{is } dA \cos\theta = dS = d\Omega \cdot R^2$$

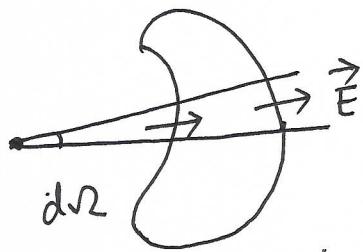
$$\Rightarrow d\Phi' = d\vec{A} \cdot \vec{E} = dA \cos\theta E$$

$$= d\Omega \cdot R^2 \cdot \frac{q}{R^2} = q d\Omega \Rightarrow d\Phi = d\Phi'$$

$$\Rightarrow \boxed{\Phi = \int d\Phi = q \int d\Omega = 4\pi q}$$



Discussion: ① For a closed surface outside the charge, $\Phi = 0$



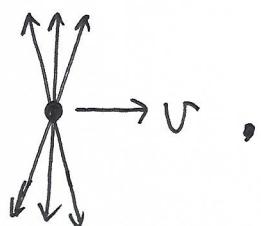
② For a many-charge system, the superposition law can be applied,

$$\begin{aligned}\Phi &= \oint \vec{E} \cdot d\vec{S} = \oint \vec{E}_1 \cdot d\vec{S} + \dots + \oint \vec{E}_n \cdot d\vec{S} \\ &= 4\pi \sum_{i=1}^n Q_i = 4\pi Q \rightarrow 4\pi \int dV \rho\end{aligned}$$

Continuous version

③ Gauss's law is consistent with the inverse square of the electric field. If $E \propto 1/R^3$, then Φ would depend on the distance and $\Phi \rightarrow 0$ as $R \rightarrow +\infty$.

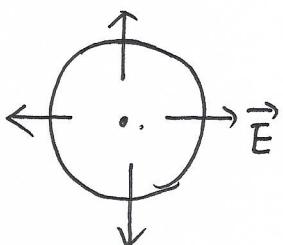
④ Gauss's law is more general than Coulomb's law. They are equivalent for electrostatics, but Gauss's law also applies to electrodynamics. For example, for a moving charge, its electric field lines exhibit as follows



But $\oint \vec{E} \cdot d\vec{S} = 4\pi q$ remain valid.

Applications: Gauss's law is a powerful method to derive the electric field distribution when combined with symmetry analysis.

① Point charge: For a spherical geometry, \vec{E} can only be along the radial direction, and exhibit the same strength at the small distance



$$\oint \vec{E} \cdot d\vec{r} = 4\pi q$$

$$E \cdot 4\pi r^2 = 4\pi q \Rightarrow E = \frac{q}{r^2}$$

or
$$\boxed{\vec{E} = \frac{q}{r^2} \hat{r}}$$

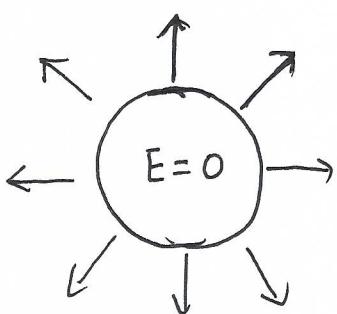
② The analysis above only involves the spherical symmetry. Hence it applies to a spherically symmetric charge distribution.



$$E \cdot 4\pi r^2 = \int_0^r r^2 \cdot dr' \cdot d\Omega \cdot p(r') \cdot 4\pi$$

$$E = \frac{Q}{r^2} = 4\pi \int_0^r p(r') r'^2 dr'$$

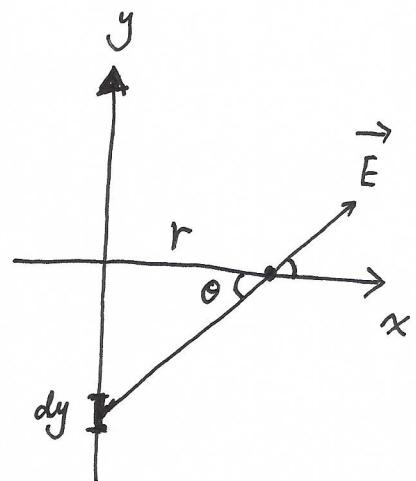
For a hollow spherical shell, the electric field inside $E=0$.



For \vec{E} field outside the shell, it looks as if the charge concentrates in the center.

Electric field of line charge

Suppose the line charge density λ . Consider a point with distance of r to the line. By the reflection symmetry with respect to x -axis, hence, \vec{E} is along x -direction.



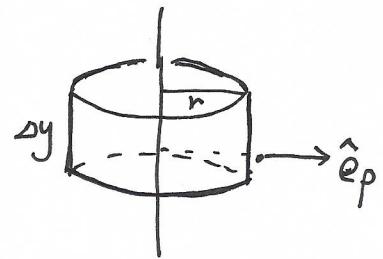
$$dE_x = \frac{\lambda \cdot dy}{(y^2 + r^2)^{3/2}} \quad r \Rightarrow E_x = \int_{-\infty}^{+\infty} \frac{\lambda dy / r}{r \left[\left(\frac{y}{r} \right)^2 + 1 \right]^{3/2}}$$

$$= \frac{\lambda}{r} \int_{-\infty}^{+\infty} dx \frac{1}{(x^2 + 1)^{3/2}}$$

Set $x = \tan \theta \Rightarrow x^2 + 1 = \sec^2 \theta$

$$\int_{-\infty}^{+\infty} dx \frac{1}{(x^2 + 1)^{3/2}} = \int_{-\pi/2}^{\pi/2} d\theta \sec^2 \theta \frac{1}{\sec^3 \theta} = \int_{-\pi/2}^{\pi/2} d\theta \cos \theta = 2$$

$$\Rightarrow E_x = \frac{2\lambda}{r} \Rightarrow \boxed{\vec{E} = \frac{2\lambda}{r} \hat{e}_x}$$



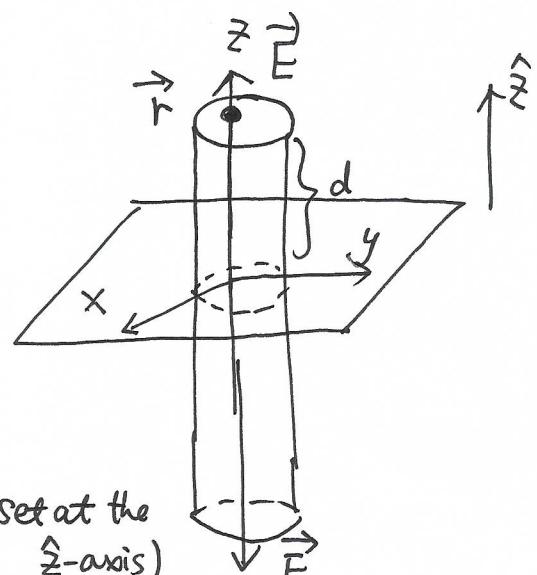
or According to Gauss's law

$$E \cdot 2\pi r \cdot dy = 4\pi dy \lambda$$

$$\Rightarrow \boxed{\vec{E} = \frac{2\lambda}{r} \hat{e}_x}$$

Electric field of planar charge

Symmetry analysis: \vec{E} -field at \vec{r} (set at the \hat{z} -axis) of a distance d above the plane,



can only be along the \hat{z} -axis

- ① The system exhibits rotational symmetry around the z-axis, hence, if \vec{E} has \hat{x} or \hat{y} components, the \vec{E} -field will not be unique.
- ② Construct a cylinder with a cross section A , and symmetric to the plane. The system has the reflection symmetry with respect to the plane, hence, $\vec{E} \parallel \hat{z}$ on the other side of the cylinder.

$$\Rightarrow E \cdot 2A = 4\pi A \sigma \rightarrow \sigma \text{ is the surface charge density}$$

$$E = 2\pi\sigma$$

E is independent on the distance to the plane!