

Lecture 4 — Electric potential

We have seen for the electric field of a point charge, ^{along} ~~any~~ closed loop ^{static}

$$\oint \vec{E} \cdot d\vec{l} = 0. \quad \left\{ \Rightarrow \nabla \times \vec{E} = 0. \right.$$

According to Stoke's theorem

Since electrostatic fields can be viewed as superposition of point charges, we have $\nabla \times \vec{E} = 0$ for any electrostatic field. Hence, \vec{E} can

be expressed as the gradient of a scalar field, i.e.

$$\vec{E} = -\nabla \varphi$$

φ : electric potential.

$$\varphi(\vec{r}_2) - \varphi(\vec{r}_1) = - \int_{\vec{r}_1}^{\vec{r}_2} d\vec{r} \cdot \vec{E}$$

φ : Electric potential of a point charge.

$$\varphi(\vec{r}) = \frac{Q}{r} \Rightarrow \vec{E} = -\nabla \varphi = \frac{Q}{r^2} \hat{r}$$

if for a charge distribution $p(x, y, z)$

$$\begin{aligned} \varphi(x, y, z) &= \int \frac{dx' dy' dz'}{|\vec{r} - \vec{r}'|} p(x', y', z') \Rightarrow \vec{E} = -\nabla \varphi \\ &= \int \frac{dx' dy' dz'}{|\vec{r} - \vec{r}'|^3} p(x', y', z') (\vec{r} - \vec{r}') \end{aligned}$$

Electric potential for an infinitely long line charge

$$\varphi(\vec{r}_2) - \varphi(\vec{r}_1) = - \int_{\vec{r}_1}^{\vec{r}_2} \vec{E} \cdot d\vec{r} = - \int_{r_1}^{r_2} \left(\frac{2\lambda}{r} \right) dr = -2\lambda \ln \frac{r_2}{r_1}$$

$$\Rightarrow \boxed{\varphi = 2\lambda \ln \frac{a}{r}}$$

where a is a constant carrying length unit.

Electric potential for a uniformly charged disk (along the \hat{z} -axis).

$$\varphi(0,0,z) = \frac{\int d\theta r dr \sigma}{R}$$

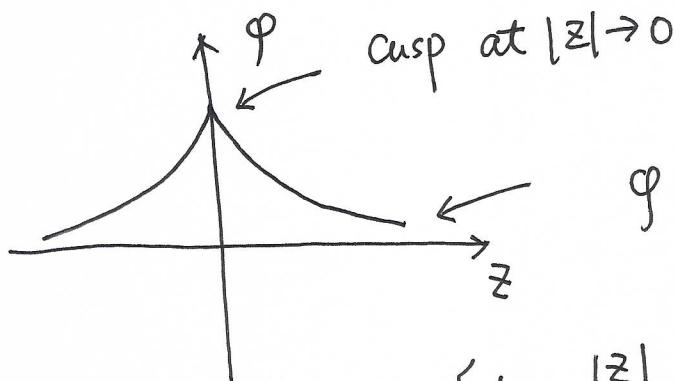
$$= 2\pi\sigma \int_0^a dr \frac{r}{\sqrt{r^2 + z^2}}$$

$$= \pi\sigma \int_0^{a^2} \frac{dr^2}{(r^2 + z^2)^{1/2}} = 2\pi\sigma (r^2 + z^2)^{1/2} \Big|_0^{a^2} = 2\pi\sigma \left[(a^2 + z^2)^{1/2} - |z| \right]$$

$$\textcircled{1} |z| \ll a, \quad \varphi(0,0,z) = 2\pi\sigma \left[a \left(1 + \frac{z^2}{za^2} \right) - |z| \right]$$

$$\approx 2\pi\sigma(a - |z|)$$

$$\textcircled{2} |z| \gg a, \quad \varphi(0,0,z) = 2\pi\sigma \left[|z| \left(1 + \frac{a^2}{2|z|^2} \right) - |z| \right] \approx \frac{\pi\sigma a^2}{|z|}$$



$\varphi \rightarrow \frac{Q}{|z|}$ at long distance.

$$\text{with } Q = \pi a^2 \sigma.$$

$$E_z = -\nabla_z \varphi(0,0,z) = 2\pi\sigma \left\{ \begin{array}{l} 1 - \frac{|z|}{\sqrt{z^2 + a^2}} \\ -2\pi\sigma \left(1 - \frac{|z|}{\sqrt{z^2 + a^2}} \right) \end{array} \right. \begin{array}{l} \text{at } z > 0, \\ \text{at } z < 0. \end{array}$$

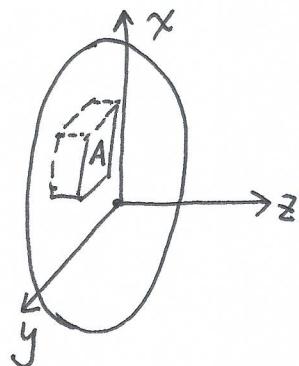
(3)

Actually, if we sufficiently approach $z \rightarrow 0$, even away from the \hat{z} -axis, the plate looks as if infinitely large. We can choose a small box

$$A \left[E_z(x, y, z=0^+) - E_z(x, y, z=0^-) \right] = 4\pi A \sigma$$

$$\Rightarrow \boxed{E_z(x, y, 0^+) - E_z(x, y, 0^-) = 4\pi \sigma}$$

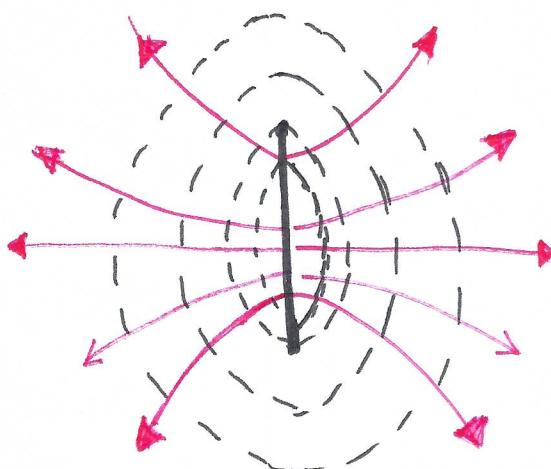
the discontinuity of the normal component of the electric field.



Since the system is reflectional symmetric with respect to the xy -plane,

$$E_z(x, y, 0^+) = -E_z(x, y, 0^-) = 2\pi \sigma.$$

The calculation for the potential and \vec{E} away from the axis in the general case is complicated. We illustrate the equal potential surfaces and electric field lines.



E -field lines are perpendicular to the equal potential surface.

§ Force on the surface charge

We know that for a uniformly charged sphere, right at the surface $E(r=R^+) = 4\pi\sigma$, and $E(r=R^-) = 0$. There's a discontinuity. If we ask which value should be used to calculate the force to the surface charge, we need to consider more.

The force ^{on area A} should be due to the electric field generated from other charges, not by the field by A itself.

The field generated by A should be

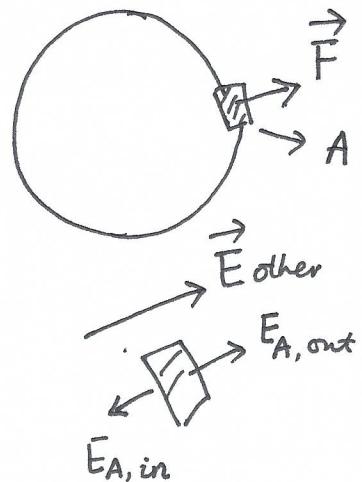
$$E_{A,\text{out}} = -E_{A,\text{in}}$$

And the field generated by other part \vec{E}_{other}

should be smooth across the shell,

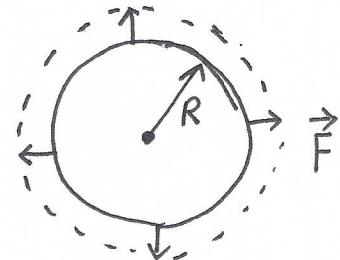
$$\Rightarrow \begin{cases} \vec{E}_{A,\text{out}} + \vec{E}_{\text{other}} = \vec{E}(R^+) \\ \vec{E}_{A,\text{in}} + \vec{E}_{\text{other}} = \vec{E}(R^-) \end{cases} \Rightarrow \vec{E}_{\text{other}} = \frac{1}{2} (\vec{E}(R^+) + \vec{E}(R^-))$$

$$\Rightarrow \vec{F} = A\sigma \vec{E}_{\text{other}} = A\sigma \cdot 2\pi\sigma \Rightarrow \boxed{\frac{dF}{dA} = 2\pi\sigma^2}$$



S Energy stored in the field

Suppose the bulb uniformly charged with Q , expand from the radius from $R \rightarrow R + \delta R$.



The environment has a pressure to balance the electrostatic force \vec{F} , such that the work done by \vec{F} is stored in the potential energy of the environment.

$$dW = F \cdot \delta R = Q \cdot 2\pi \sigma \cdot \delta R = \frac{Q^2}{2R^2} \delta R.$$

On the other hand, the only difference of the expansion is the change of electric field within the volume of the shell.

$$E = \frac{Q}{R} \Rightarrow dW = \frac{E^2 \cdot \pi R^2}{2R^2} \delta R = \frac{E^2}{8\pi} 4\pi R^2 \delta R$$

$$\Rightarrow dW = \frac{E^2}{8\pi} \delta V$$

This means that the electric

potential energy, which switches to the

mechanical energy, is originally stored in the electric field!

Theorem: The electric potential energy U stored in a charge system

can be expressed as

$$U = \frac{1}{8\pi} \int dV E^2$$

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$$E = -\nabla \varphi \quad \Rightarrow \quad \nabla \cdot \vec{E} = -\nabla^2 \varphi$$

$$\Rightarrow U = \frac{1}{8\pi} \int E^2 dV = \frac{1}{8\pi} \int (\nabla \varphi)^2 dV$$

$$(\nabla \varphi)(\nabla \varphi) = \nabla \cdot (\varphi \nabla \varphi) - \varphi \nabla^2 \varphi$$

Gauss's law: $\oint \vec{E} \cdot d\vec{s} = \iiint_V \nabla \cdot \vec{E} = \iiint_V \rho \cdot 4\pi$

$$\Rightarrow \boxed{\nabla \cdot \vec{E} = -\nabla^2 \varphi = 4\pi \rho} \quad \rightarrow \text{Poisson Eq.}$$

$$\Rightarrow U = \frac{1}{8\pi} \iint \nabla \cdot (\varphi \nabla \varphi) dV + \frac{1}{2} \iiint_V \rho \varphi$$

$$= \frac{1}{8\pi} \iint (\varphi \cdot \nabla \varphi) d\vec{s} + \frac{1}{2} \iiint_V \rho \varphi dV$$

$$\varphi \sim \frac{1}{R}, \quad \nabla \varphi \sim \frac{1}{R^2}, \quad \text{hence} \quad \varphi \nabla \varphi \sim \frac{1}{R^3}$$

if we set $R \rightarrow +\infty$, then the boundary term $\rightarrow 0$.

$$U = \frac{1}{2} \iiint_V \rho \varphi dV$$

\Rightarrow for static system.
electro