

## Lect 5 : Electric fields around conductors

Insulators: bound electrons. Under an external  $\vec{E}$  field, electrons can be displaced, and acquire  $a$  new equilibrium. No current!

Conductor: electrons acquire a finite velocity, developing current.

Electron can travel to an finite distance within a finite time!

The crucial difference between insulators and conductors can only be explained by quantum mechanics.

### § Conductors under $\vec{E}$ -field

Imagine applying  $\vec{E}_{ex}$  field to a conductor.

Electrons are driven to redistribute resulting a counter-electric field

which balances the external field  $\vec{E}_{ex}$ .

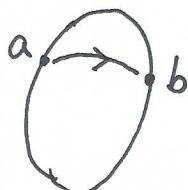
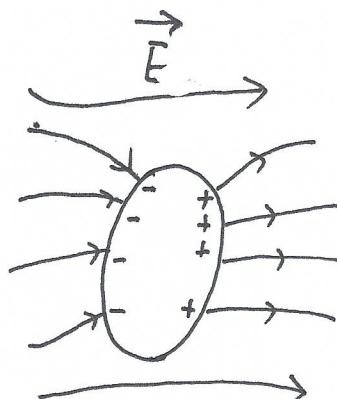
Only when  $\vec{E}$  inside of metal becomes zero, electrons stop moving.

Otherwise, more electrons redistribute to cancel  $\vec{E}$ . Finally  $\vec{E}$  field becomes zero. Then the surface of metal becomes a iso-potential

surface:

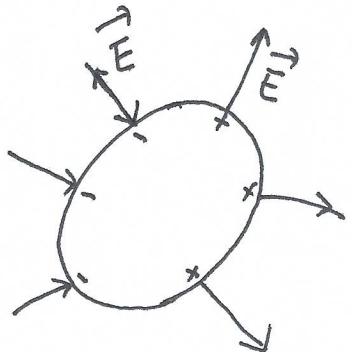
①

$$\varphi_b - \varphi_a = \int_a^b \vec{dr} \cdot \vec{E} = 0.$$



②  $\vec{E}$ -field perpendicular to the surface of a metal.

$$\vec{E} = -\nabla U, \text{ hence, } \vec{E} \perp \text{metal surface.}$$



Then the surface charge density

$$E_{n,out} - E_{n,in} = 4\pi \sigma$$

$$\Rightarrow E_{n,out} = 4\pi \sigma$$

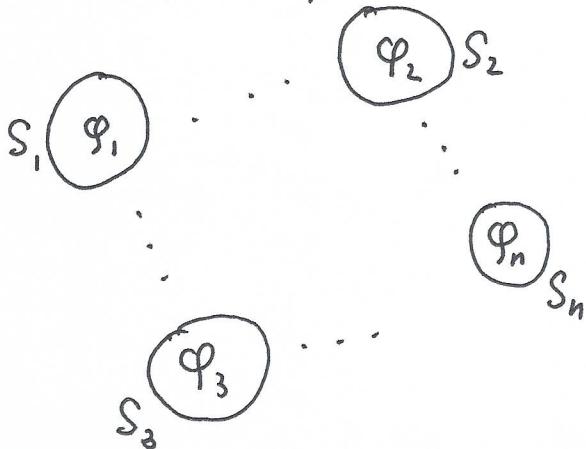
$E_{n,out} > 0$  if  $\vec{E}$  points outside  
 $< 0$  points inside.

Hence, the total charge of the metal

$$Q = \oint \sigma d\alpha = \frac{1}{4\pi} \oint \vec{E} \cdot d\vec{\alpha}$$

### { Uniqueness of the electrostatic problem

Suppose there exist a few metallic surfaces with the electric potential  $\varphi_1, \dots, \varphi_n$ , respectively, then the distribution of electric potential is uniquely determined!  
(No other charges in space.)



not mathematically  
physically, the existence of  
the solution is out of  
question.

Below we focus on the  
uniqueness!

(3)

The solution of the electric potential  $\varphi$  satisfies

$$\nabla \cdot \vec{E} = -\nabla \cdot (\nabla \varphi) = -\nabla^2 \varphi = 4\pi \rho.$$

We only consider the case without free charges, then the electric potential satisfies the Laplace equation!

$$\left\{ \begin{array}{l} \nabla^2 \varphi = 0, \text{ with the boundary condition} \\ \varphi|_{S_i} = \varphi_i \quad \text{for } i=1,2,\dots,n. \end{array} \right.$$

Suppose there exist two different solutions  $\varphi$  and  $\psi$ . Then

define  $W = \varphi - \psi$ , such that  $\left\{ \begin{array}{l} \nabla^2 W = 0 \\ W|_{S_i} = 0 \text{ for } i=1,\dots,n. \end{array} \right.$

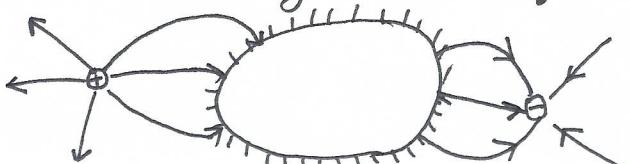
Then  $\int (\nabla \cdot W)^2 d^3 r = \int (\nabla \cdot (W \nabla W) - W \nabla^2 W) d^3 r$

$$= \int \nabla \cdot (W \nabla W) d^3 r = \oint_{\text{boundary of } S_i} W \cdot \nabla W d\vec{s} = 0$$

$$\Rightarrow \nabla \cdot W = 0 \Rightarrow \varphi - \psi = \text{const},$$

$$\left\{ W|_{S_i} = 0 \right. \Rightarrow \boxed{\varphi = \psi}$$

- For a hollow conductor with any shape, if there's no charge inside, the  $\vec{E}$  inside the conductor equals zero.

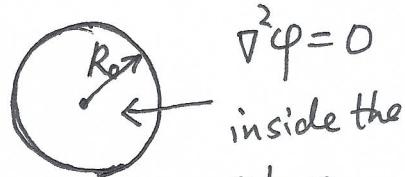


- An important property of harmonic function  $\nabla^2 \varphi = 0$ .

Draw a sphere with a finite radius  $R$ , then the average of  $\varphi(\vec{r})$

on the sphere equals the value of  $\varphi$  located at the center of the sphere.

$$\int dS \cdot \varphi = 4\pi R^2 \varphi(\vec{r}=0).$$



Proof: We design a charge density distribution

$$\rho = \sigma \delta(r - R_0). \text{ Then the charge distributes on the surface}$$

$$\text{with total charge } Q = \int r^2 dr \rho \cdot 4\pi = 4\pi R_0^2 \sigma.$$

Assume the potential generated by  $\rho(\vec{r})$  as  $\bar{\Psi}$ , i.e.  $\nabla^2 \bar{\Psi} = -4\pi \rho(\vec{r})$ .

We calculate the electric potential energy

$$\int \rho \varphi d^3 \vec{r} = -\frac{1}{4\pi} \int d^3 \vec{r} \nabla^2 \bar{\Psi}(\vec{r}) \varphi(\vec{r}).$$

$$\text{According to } \nabla^2 \bar{\Psi} \cdot \varphi = \nabla(\nabla \bar{\Psi} \cdot \varphi) - \nabla \bar{\Psi} \cdot \nabla \varphi = \nabla(\nabla \bar{\Psi} \cdot \varphi - \bar{\Psi} \cdot \nabla \varphi) + \bar{\Psi} \cdot \nabla^2 \varphi.$$

$$\Rightarrow \int \rho \varphi d^3 \vec{r} = -\frac{1}{4\pi} \oint_{\text{boundary}} (\nabla \bar{\Psi} \cdot \varphi - \bar{\Psi} \cdot \nabla \varphi) dS - \frac{1}{4\pi} \int d^3 \vec{r} \bar{\Psi} \cdot \nabla^2 \varphi$$

Here the surface integral over boundaries, which are outside the sphere we are investigating.  $\nabla^2 \varphi = -4\pi \rho'(\vec{r})$ , where  $\rho'(\vec{r})$  also distribute outside the sphere. Hence, the contribution to the RHS

are determined by  $\Psi$  outside the sphere with the radius  $R_0$ .

$$4\pi R_0^2 \sigma \langle \varphi \rangle \Big|_{\substack{\text{Sphere} \\ \text{with the radius } R_0}} = -\frac{1}{4\pi} \oint \oint (\nabla \Psi \cdot \varphi - \Psi \cdot \nabla \varphi) \cdot d\vec{S} + \int d^3 r \Psi \rho(\vec{r}) \Big|_{\substack{\text{boundary} \\ \text{outside the sphere}}} \Big|_{\substack{\text{outside} \\ \text{the sphere}}}.$$

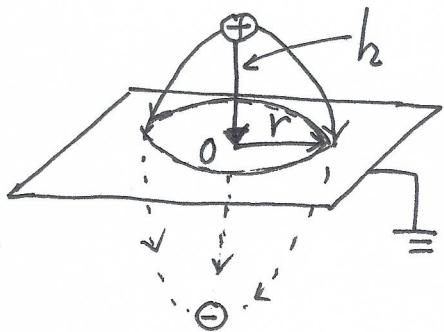
We know that  $\Psi$  is generated by  $\rho(\vec{r}) = \sigma \delta(r - R_0)$ . Its distribution outside the sphere is the same if all the charge concentrates to the center according to Gauss' law. Then the LHS is independent on the value of  $R_0$ , as long as the sphere does not include any other charge, i.e.

$$4\pi R_0^2 \sigma \langle \varphi \rangle \Big|_{\substack{\text{sphere with radius } R_0}} = Q \varphi(\vec{r} = \text{center})$$

$$\Rightarrow \boxed{\langle \varphi \rangle \Big|_{\substack{\text{sphere with the radius } R_0}} = \varphi(\vec{r} = \text{center})}$$

Application of the uniqueness theorem:

Consider a positive charge  $+q$  put above a grounded conducting plate. According to the uniqueness theorem, we can design an equivalent system (but much simpler) which provides the same boundary condition.



Then we replace the grounded plates by a negative charge  $-q$ , then the bisector plane is the zero potential plane.

Consider a point on the plate with the distance  $r$  from the projection of the  $\oplus-\ominus$  line  $O$ . Then the field is along the  $z$ -axis

$$E_z = -2 \frac{+q}{(r^2+h^2)} \frac{h}{(r^2+h^2)^{1/2}} = -\frac{2qh^2}{(r^2+h^2)^{3/2}}$$

$$\text{hence } \sigma_z = \frac{E_z}{4\pi} = -\frac{1}{2\pi} \frac{2hq}{(r^2+h^2)^{3/2}}$$

$$\text{then the total charge } Q = \int \sigma_z r^2 dr d\theta = \frac{2\pi Q}{2\pi} \int_0^\infty r dr \frac{h}{(r^2+h^2)^{3/2}}$$

$$= -q \int_0^\infty \frac{r}{h} \frac{dr}{((r/h)^2+1)^{3/2}} = -q \int_0^\infty \frac{x dx}{(x^2+1)^{3/2}}$$

$$= -q.$$

# Capacitors

- energy stored in capacitors

$$U = \frac{Q}{C} \Rightarrow E = \int_0^{Q_f} u dQ = \frac{1}{C} \int_0^{Q_f} Q dQ$$

$$= \frac{1}{2C} Q_f^2$$

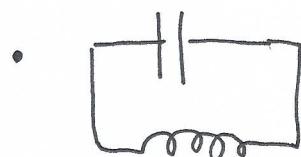
$$E = \frac{1}{2C} (Cu_f)^2 = \frac{C}{2} u_f^2$$

or The force on a plate, say, the upper one.  $F = \frac{E_{in} + E_{out}}{2} \cdot Q = \frac{2\pi\sigma}{2} \cdot Q$

$$\Rightarrow U = F \cdot d = 2\pi \frac{Q^2}{A} \cdot d = \frac{Q^2}{2} \cdot \frac{4\pi \cdot d}{A} \quad \left. \begin{array}{l} \\ \end{array} \right\} \Rightarrow U = \frac{Q^2}{2C}$$

$$C = \frac{A}{4\pi d}$$

The latter method is consistent with  $\mathcal{E}/V = \frac{E^2}{8\pi}$ , (the energy density stored in the electric field).



L-C circuit

$$E = \frac{Q^2}{2C} + \frac{1}{2} L \left( \frac{dQ}{dt} \right)^2$$

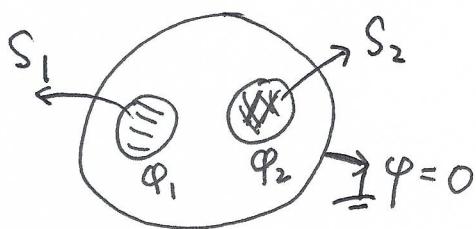
$$= \frac{Q^2}{2C} + \frac{\Phi^2}{2L} \quad \text{where } \Phi = LI$$

$$= L \frac{dQ}{dt}$$

Analogy to a mechanical oscillator:

$$L \rightarrow m, \quad C^{-1} \rightarrow m\omega^2 \Rightarrow \omega = \sqrt{\frac{1}{LC}}$$

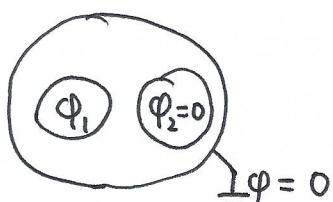
• Capacitances of a group of conductors / charges



① two metallic balls with the electric potentials  $\varphi_1$  and  $\varphi_2$ , with the outer shell grounded

The relation between  $Q_{1,2}$  and  $\varphi_{1,2}$  are  $\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$  ?

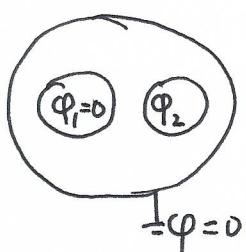
② The above relation can be derived via the superposition principle.



$$Q_1^I = C_{11} \varphi_1$$

$$Q_2^I = C_{21} \varphi_1$$

case I

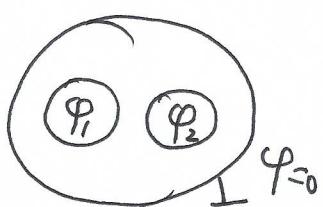


case II

$$Q_1^{II} = C_{12} \varphi_2$$

$$Q_2^{II} = C_{22} \varphi_2$$

Case I + Case II



$$\left\{ \begin{array}{l} \nabla^2 \varphi_I = 0 \\ \varphi_I = \varphi_1 |_{S_1} \\ \quad 0 |_{S_2} \\ \quad 0 |_{\text{outshell}} \end{array} \right.$$

$$\left\{ \begin{array}{l} \nabla^2 \varphi_{II} = 0 \\ \varphi_{II} = 0 |_{S_1} \\ \quad \varphi_2 |_{S_2} \\ \quad 0 |_{\text{outshell}} \end{array} \right.$$

$$\Rightarrow \nabla^2 (\varphi_I + \varphi_{II}) = 0$$

$$\Rightarrow Q_1 = Q_1^I + Q_1^{II}$$

$$\varphi_I + \varphi_{II} = \begin{cases} \varphi_1 & |_{S_1} \\ \varphi_2 & |_{S_2} \\ 0 & |_{\text{outshell}} \end{cases}$$

$$Q_2 = Q_2^I + Q_2^{II}$$