

## Lect 16. Special Relativity (II)

§1 the relativistic velocity addition

$$\left( \begin{matrix} dx' \\ ct' \end{matrix} \right) = \left( \begin{matrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{matrix} \right) \left( \begin{matrix} dx \\ ct \end{matrix} \right) \Rightarrow v_x' = \frac{dx'}{dt'} = \frac{\gamma dx - \gamma\beta \frac{c}{c}}{-\gamma\beta \frac{dx}{c} + \gamma dt} = \frac{\frac{dx}{dt} - \beta}{-\beta \frac{dx}{dt} + 1}$$

$$= \frac{v_x - \nu}{1 - \frac{v_x \nu}{c^2}}$$

$$v_y' = \frac{dy'}{dt'} = \frac{dy}{-\gamma\beta \frac{dx}{c} + \gamma dt} = \frac{v_y}{\gamma(1 - \frac{v_x \nu}{c^2})}, \quad v_z' = \frac{v_z}{\gamma(1 - \frac{v_x \nu}{c^2})}.$$

if  $v_x = c \Rightarrow v_x' = \frac{c - \nu}{1 - \frac{\nu c}{c^2}} = c$ , light velocity doesn't change!  
 $v_y = v_z = 0$

$$\text{if } v_x, \nu < c \quad \frac{v_x'}{c} = \frac{\frac{v_x}{c} - \frac{\nu}{c}}{1 - \frac{v_x \nu}{c^2}} = \frac{\beta_x - \beta}{1 - \beta_x \beta},$$

$$(1 - \beta_x)(1 + \beta) > 0 \Rightarrow 1 - \beta_x \beta > \beta_x - \beta \quad \} \Rightarrow 1 - \beta_x \beta > |\beta_x - \beta|$$

$$(1 + \beta_x)(1 - \beta) > 0 \Rightarrow 1 - \beta_x \beta > \beta - \beta_x$$

$$\Rightarrow \left| \frac{v_x'}{c} \right| < 1.$$

## §2. 4-vector

$$x^\mu = (x_1, x_2, x_3, ct), \rightarrow \text{inner product } x^\mu x_\mu = x_1^2 + x_2^2 + x_3^2 - (ct)^2$$

$$x_\mu = (x_1, x_2, x_3, -ct) \quad x_\mu = g_{\mu\nu} x^\nu, \quad x^\mu = g^{\mu\nu} x_\nu \quad g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$$

4-vector  $q^\mu$ , transforms as  $q^\mu = \Lambda^\mu_\nu q^\nu$ .

$$(q_1, q_2, q_3, q_4)$$

$\Lambda^\mu_\nu$  is the general Lorentz transformation,

6-degrees of freedom.

$$\text{satisfying } \Lambda^T g \Lambda = g.$$

$$\text{or } \Lambda^{\mu'}_\mu g_{\mu\nu} \Lambda^\nu_\nu = g_{\mu\nu}.$$

$$\Lambda: \begin{bmatrix} & & 0 \\ R & & 0 \\ - & - & - \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} \text{spatial part: 3d rotation} \\ \text{3-degrees of freedom} \end{array}$$

$$\begin{bmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{bmatrix} \quad \begin{array}{l} \text{space-time boost along } x, y, z\text{-directions} \\ \text{3-degrees freedom.} \end{array}$$

the inner product of 4-vectors is invariant under Lorentz transform

$$q_1^\mu q_{\mu\nu} q_2^\nu = q_1^\mu \underbrace{\Lambda^\mu_{\mu'} g_{\mu\nu}}_{g_{\mu\nu'}} \Lambda^\nu_{\nu'} q_2^{\nu'} = q_1^{\mu'} g_{\mu'\nu'} q_2^{\nu'}.$$

4-velocity

$$u^\mu = \frac{dx^\mu}{dt} = \left[ \frac{1}{\sqrt{1-\beta^2}} \frac{dx}{dt}, \frac{1}{\sqrt{1-\beta^2}} \frac{dy}{dt}, \frac{1}{\sqrt{1-\beta^2}} \frac{dz}{dt}, \frac{c}{\sqrt{1-\beta^2}} \right]$$

### { Doppler Effect :

quotient rule: if  $(\vec{x}, x_4)$  is a 4-vector, and  $(\vec{k}, k_4)$  satisfies

$\vec{k} \cdot \vec{x} - k_4 x_4$  is invariant in any frame (Scalar), then  $(\vec{k}, k_4)$  is a 4-vector.

Suppose two frames  $S$  and  $S'$

$$\underbrace{k^{\mu'} g_{\mu\nu'} x^{\nu'}} = k'^{\mu} g_{\mu\nu} x^{\nu} = \underbrace{k'^{\mu} g_{\mu\nu} \Lambda^{\nu}_{\nu'} x^{\nu'}}$$

$$\Rightarrow k^{\mu'} g_{\mu\nu'} = k'^{\mu} g_{\mu\nu} \Lambda^{\nu}_{\nu'} \quad g_{\mu\nu'} = \Lambda^{\mu}_{\mu'} g_{\mu\nu} \Lambda^{\nu}_{\nu'}$$

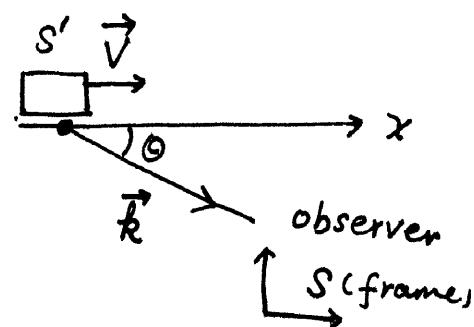
$$k^{\mu'} \Lambda^{\mu}_{\mu'} g_{\mu\nu} \Lambda^{\nu}_{\nu'} = k'^{\mu} g_{\mu\nu} \Lambda^{\nu}_{\nu'} \Rightarrow \boxed{k'^{\mu} = \Lambda^{\mu}_{\mu'} k^{\mu'}}$$

$(\vec{k}, k_4)$  is also a 4-vector.

$\phi = A \cos(\vec{k} \cdot \vec{x} - \omega t)$ , the phase factor  $\vec{k} \cdot \vec{x} - \omega t = k^{\mu} g_{\mu\nu} x^{\nu}$

$\Rightarrow k^{\mu} = (\vec{k}, \frac{\omega}{c})$  is a 4-vector.

In the co-moving frame  $S' \Rightarrow k'_4 = \frac{\omega_0}{c}$ .



$$k'_4 = \gamma(k_4 - \beta k_1) = \frac{\omega_0}{c}$$

$$k_1 = |k| \cos \theta = \frac{\omega}{c} \cos \theta$$

$$k_4 = \frac{\omega}{c}$$

$$\Rightarrow \gamma \frac{\omega}{c} [1 - \beta \omega s \theta] = \frac{\omega_0}{c}$$

$$\omega = \frac{1}{\gamma} \frac{\omega_0}{1 - \beta \omega s \theta}$$

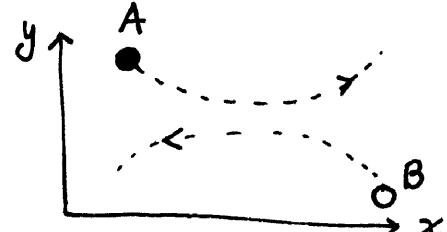
at  $\theta = 0, \pi$   
 $\omega = \frac{1}{\gamma} \frac{\omega_0}{1 + \beta}$

at  $\theta = \frac{\pi}{2}$   
 $\omega = \frac{\omega_0}{\gamma}$

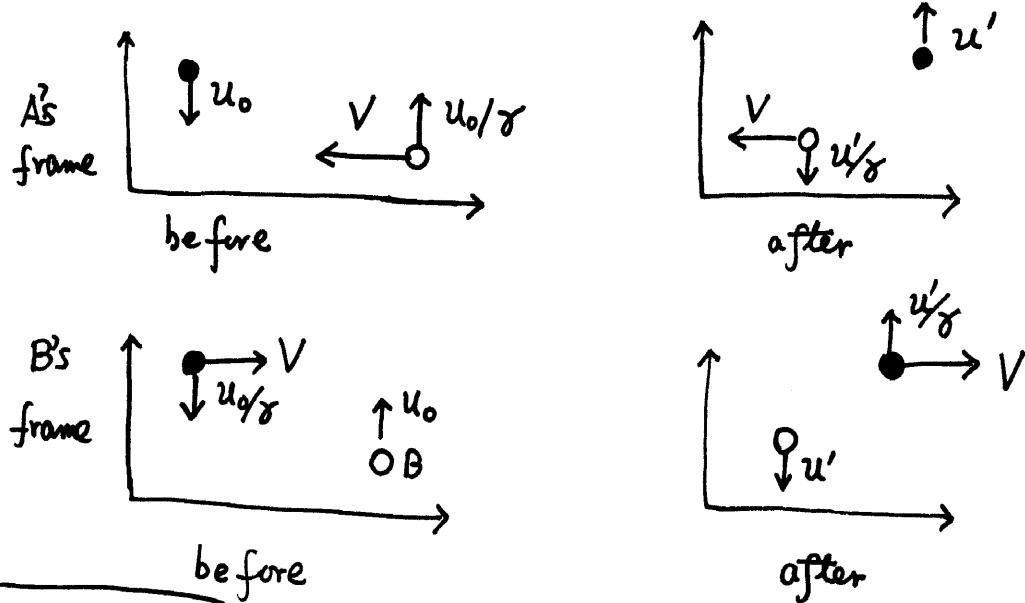
## § relativistic momentum

We want to maintain momentum conservation law. But the classic definition  $\vec{P} = m_0 \vec{v}$  does not work any more. We define "m" the rest mass measured in the frame where the particle is static.

Consider the following collision process



This collision process in the frames moving with the same horizontal velocity as A, B, respectively.



### Before Collision:

in B's frame, B's horizontal velocity is zero, vertical velocity  $u_0$ .

then in A's frame which has a relative  $V$  respect to B's.

$$\text{then } V_{B,x} = -V, \quad V_{B,y} = \frac{u_0}{\sqrt{1 - (\frac{V}{c})^2}} = \frac{u_0}{\gamma}$$

$$V_{A,x} = 0, \quad V_{A,y} = -u_0$$

After collision:

$$\text{in A's frame: } V'_{A,x} = 0, \quad V'_{A,y} = u'$$

$$V'_{B,x} = -v, \quad V'_{B,y} = \frac{u'}{\gamma}, \quad \text{where } \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

We seek a conserved quantity similar to classic momentum. We

define  $\vec{P} = m(v) \vec{V}$ , where  $m(v)$  is a function of  $v$ , and has the unit of mass.

In A's frame, the  $P_x$  comes from B. Before collision

$$v_B = \left( v^2 + \frac{u_0^2}{\gamma^2} \right)^{1/2}, \quad \text{and after collision } v'_B = \left( v^2 + \frac{u'^2}{\gamma^2} \right)^{1/2}$$

$$P_x^{\text{before}} = P_x^{\text{end}} \Leftrightarrow m(v_B) v = m(v'_B) v$$

$$\Rightarrow u_0 = u'.$$

Now let's write the momentum conservation along y-axis.

$$-m(u_0) u_0 + m\left(\sqrt{v^2 + \frac{u_0^2}{\gamma^2}}\right) \frac{u_0}{\gamma} = m(u_0) u_0 - m\left(\sqrt{v^2 + \frac{u'^2}{\gamma^2}}\right) \frac{u_0}{\gamma}$$

$$\Leftrightarrow m\left(\sqrt{v^2 + \frac{u_0^2}{\gamma^2}}\right) = \gamma m(u_0)$$

set  $u_0 \rightarrow 0$ , we have  $m(v) = \gamma m_0 = \gamma m_0$   
 $\uparrow$  rest mass.

thus we define  $\vec{P} = \frac{m_0 \vec{v}}{\sqrt{1 - v^2/c^2}}$  which is conserved.

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$$\text{check } m \left( \sqrt{v^2 + \frac{u_0^2}{c^2}} \right) = \gamma m(u_0)$$

$$\begin{aligned} \frac{1}{\sqrt{1 - \left( \frac{v^2 + u_0^2}{c^2} \right)}} &= \frac{\gamma}{\sqrt{\gamma^2 - \left( \frac{v^2 + u_0^2}{c^2} \right)}} = \frac{\gamma}{\sqrt{\gamma^2 \left( 1 - \frac{v^2}{c^2} \right) - \frac{u_0^2}{c^2}}} \\ &= \frac{\gamma}{\sqrt{1 - \frac{u_0^2}{c^2}}} \quad \checkmark \text{ correct!} \end{aligned}$$

### § Relativistic energy

we generalize newton's second law  $\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt} \left[ \frac{m_0 \vec{v}}{\sqrt{1 - \frac{v^2}{c^2}}} \right]$

define kinetic energy  $E_K(v)$

$$E_K(v_b) - E_K(v_a) = \int_a^b \vec{F} \cdot d\vec{r} = \int_a^b \frac{d\vec{p}}{dt} \cdot d\vec{r}$$

$$= \int_a^b \frac{d}{dt} \left( m_0 \frac{\vec{v}}{\sqrt{1 - \frac{v^2}{c^2}}} \right) \cdot \vec{v} = \frac{m_0 v^2}{\sqrt{1 - \frac{v^2}{c^2}}} \Big|_a^b - \int_a^b \frac{m_0 \vec{v}}{\sqrt{1 - \frac{v^2}{c^2}}} d\vec{v}$$

$$\vec{v} \cdot d\vec{v} = \frac{1}{2} d(\vec{v} \cdot \vec{v}) = \frac{1}{2} d v^2 = v dv$$

$$= \frac{m_0 v^2}{\sqrt{1 - \frac{v^2}{c^2}}} \Big|_a^b - \int_a^b \frac{m_0 v}{\sqrt{1 - \frac{v^2}{c^2}}} dv = \frac{m_0 v^2}{\sqrt{1 - \frac{v^2}{c^2}}} \Big|_a^b + m_0 c^2 \sqrt{1 - \frac{v^2}{c^2}} \Big|_a^b$$

total energy

Set  $v_a = 0$

$$\Rightarrow E_K = \frac{m_0 v_b^2}{\sqrt{1 - \frac{v_b^2}{c^2}}} + m_0 c^2 \sqrt{1 - \frac{v_b^2}{c^2}} - m_0 c^2 = \frac{m_0 c^2}{\sqrt{1 - \frac{v_b^2}{c^2}}} - m_0 c^2$$

↑ rest energy

more general than just mechanical energy!

$$E = mc^2$$

## S 4-momentum

We can combine momentum and energy as a 4-vector.

$$P^\mu = \left( \vec{P}, \frac{m_0 c^2}{\sqrt{1 - \frac{v^2}{c^2}}} \right) = \left( \frac{m_0 \vec{x}/dt}{\sqrt{1 - \frac{v^2}{c^2}}}, \frac{m_0 c^2}{\sqrt{1 - \frac{v^2}{c^2}}} \right) = m_0 \left( \frac{d\vec{x}}{d\tau}, \frac{cdt}{d\tau} \right)$$

$$d\tau = \sqrt{1 - \frac{v^2}{c^2}} dt$$

$d\tau$  is the proper time interval for a moving body. 4-momentum is just rest-mass times 4-velocity.

\* For a relativistic collision, all the 4-component must be conserved.

Let us look at the 4th component, and consider the non-relativistic limit.

$$E_a^{in} + E_b^{in} = E_a^{fin} + E_b^{fin} \quad \text{expand to second order of } v$$

$$m_{a,0}^{in} c^2 + \frac{1}{2} m_{a,0}^{in} v_a^2 + (a \rightarrow b) = m_{a,0}^{fin} c^2 + \frac{1}{2} m_{a,0}^{fin} v_a^{fin^2} + (a \rightarrow b)$$

$$M_0^{in} c^2 + T^{in} = M_0^{fin} c^2 + T^{out}$$

for elastic collision,  $T^{in} = T^{out} \Rightarrow \cancel{\text{mass}}$  are conserved

in elastic collision.  $M^{in} \neq M^{fin}$ . particles can gain mass by gaining internal energy.

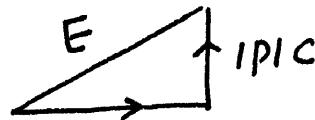
\* useful relations  $(\vec{P}, E/c)$

$$\beta = \frac{v}{c} = \frac{\vec{P}c}{E}, \quad P^2 - \frac{E^2}{c^2} = -(mc)^2 \quad \text{check } (0, 0, 0, mc)$$

$$E^2 = (pc)^2 + (mc^2)^2$$

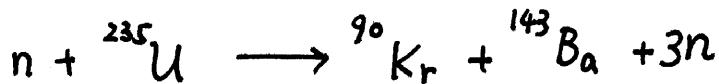
$$\uparrow \quad mc_e^2 = 0.5 \text{ Mev, if } T = 0.8 \text{ Mev} \quad mc^2$$

\* Example  $\Rightarrow E = 1.3 \text{ Mev}$  and  $pc = 1.2 \text{ Mev} \Rightarrow \beta = \frac{pc}{E} = \frac{1.2}{1.3} \approx 0.92.$



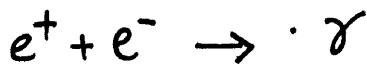
- ① Frank-Hertz experiment: inelastic collision between electron and Hg atom. Electron loses kinetic energy 4.9 eV (it's rest mass does not change since it's a point particle). This energy is transferred into Hg atom to an excited state. The mass of Hg increase  $\Delta M = \frac{4.9 \text{ eV}}{c^2} = 8.7 \times 10^{-36} \text{ kg}$   
rest
- $$\frac{\Delta M}{M_{\text{Hg}}} \approx 2.6 \times 10^{-11}.$$

- ② neutron-induced fission



$$\Delta T_{\text{kinetic}} = 200 \text{ Mev}, \quad \frac{\Delta M}{M} \approx 0.1\%$$

- ③ particle - anti-particle annihilation



(This process can not occur in ~~vacuum~~ free space because of momentum conservation, but can occur for an  $e^+$  collides with an atom).

## Lect 17. Relativity (III)

§ Collisions — application of momentum-energy conservation.

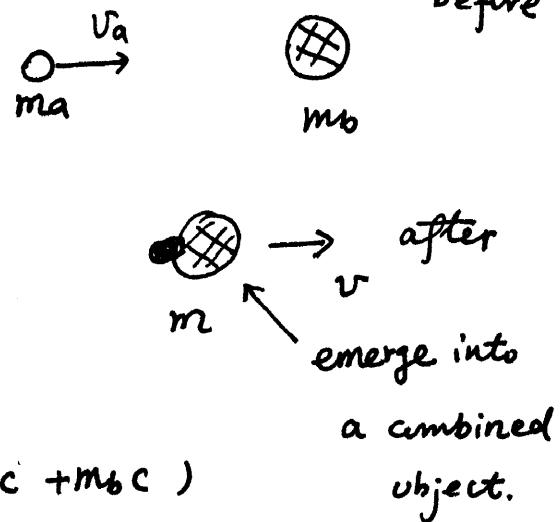
Example 1:

$$P_{a,i}^{\mu} = (m_a \gamma_a v_a, 0, 0, m_a \gamma_a c^{-2})$$

$$P_{b,i}^{\mu} = (0, 0, 0, m_b c^{-2})$$

$$P_f = (m \gamma v, 0, 0, m \gamma c^2)$$

$$= P_{a,i} + P_{b,i} = (m_a \gamma_a v_a, 0, 0, m_a \gamma_a c + m_b c)$$



$$\Rightarrow m^2 c^2 \gamma^2 \left[ 1 - \left( \frac{v}{c} \right)^2 \right] = (m_a \gamma_a c + m_b c)^2 - m_a \gamma_a v_a^2$$

$$(mc)^2 = (m_a c)^2 + (m_b c)^2 + 2m_a m_b \gamma_a c \cdot v^2$$

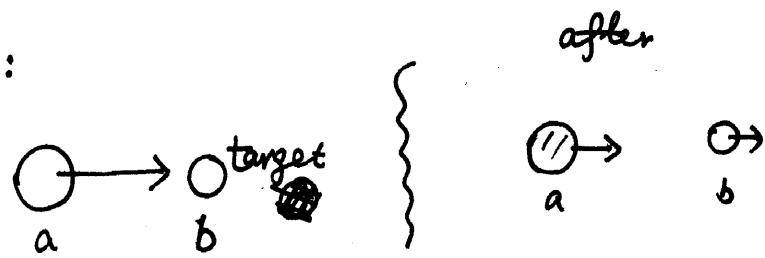
$$m = (m_a^2 + m_b^2 + 2m_a m_b \gamma_a)^{1/2} > (m_a + m_b)$$

$$\frac{\vec{v}}{c} = \frac{m \gamma v}{m \gamma c} = \frac{m_a \gamma_a v_a}{(m_a \gamma_a + m_b)c} \Rightarrow v = \frac{m_a \gamma_a}{m_a \gamma_a + m_b} v_a$$

Center of mass frame (zero momentum frame).

Example 2: elastic head on collision

Lab frame S:



in the CM frame S':



In the lab frame, the final state will be difficult to calculate.

But in the CM frame, they just reverse the direction. (simple!).

Let us find the relation between lab frame & CM frame.

• Lab frame: total momentum

$$P_{\text{tot}}^L = (\vec{P}_a, \frac{E_a}{c}) + (0, m_b c) = (\vec{P}_a, \frac{E_a + m_b c^2}{c})$$

in the C-M frame which has relative velocity  $\beta c$  respect to Lab frame

$$P'_1 = \gamma(P_a - \beta \frac{E_a + m_b c^2}{c}) = 0 \Rightarrow \beta = \frac{P_a c}{E_a + m_b c^2}$$

In this frame, the final state of  $b$  can be obtained from

the  $P_b = (0, m_b c)$  in the Lab frame

initial  
frame

$$P'_{b,\text{in},1} = \gamma(0 - \beta m_b c) = -\gamma m_b c \quad \left. \begin{array}{l} \beta \\ \gamma \end{array} \right\} \Rightarrow 1$$

$$P'_{b,\text{in},4} = \gamma(-\beta \cdot 0 + m_b c) = \gamma m_b c \quad \left. \begin{array}{l} \beta \\ \gamma \end{array} \right\}$$

(3)

$\Rightarrow$  in the CM frame.  $P_{\text{final}}^b = (\gamma \beta m_b c, \gamma m_b c)$ .

back to Lab frame

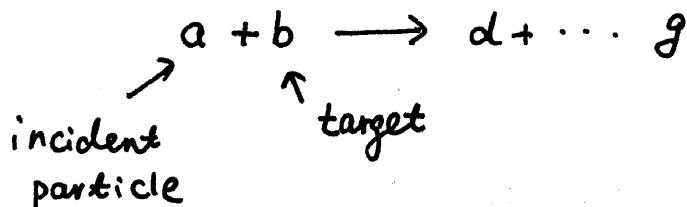
$$P_{1, \text{final}}^b = \gamma (\gamma P'_{1, \text{final}} + \beta P'_{4, \text{final}}) = \gamma [\gamma \beta + \gamma \beta] m_b c \\ = 2 \gamma^2 \beta m_b c$$

$$P_{4, \text{final}}^b = \gamma [\beta P'_{1, \text{final}} + P'_{4, \text{final}}] = \gamma [\gamma \beta^2 + \gamma] m_b c \\ = \gamma^2 [\beta^2 + 1] m_b c$$

$\Rightarrow$  the final velocity of B

$$\boxed{\frac{P_1 c}{P_4} = \frac{2 \gamma^2 \beta}{\gamma^2 (\beta^2 + 1)} c = \frac{2 \beta}{1 + \beta^2} c}$$

{ threshold energies.



which is the minimum energy of  $a$  in the lab frame to create  $d, \dots g$ .

Lab frame  $P_{\text{tot}}^{\mu} = (P_a, E_a/c) + (0, m_b c) = (P_a, \frac{E_a + m_b c^2}{c})$

$\rightarrow$  ~~Lab~~ frame  $\beta = \frac{P_a c}{E_a + m_b c^2}$   
Center of M

(4)

in this frame: ~~as particle~~ the total energy

$$\cancel{P_\alpha} \cancel{\frac{E_\alpha}{c}} \quad (E_{cm, \text{tot}}/c)^2 = \left( \frac{E_a + m_b c^2}{c} \right)^2 - P_\alpha^2$$

$$\Rightarrow E_{cm}^2 = E_a^2 + m_b^2 c^4 + 2 E_a m_b c^2 - P_\alpha^2 c^2 = (m_a^2 + m_b^2) c^4 + 2 E_a m_b c^2$$

$$E_{cm} \geq \sum m_{fin} c^2$$

$$\Rightarrow m_a^2 c^4 + m_b^2 c^4 + 2 E_a m_b c^2 \geq (\sum m_{fin})^2 c^4$$

$$\Rightarrow E_a \geq \boxed{\frac{(\sum m_{fin})^2 - m_a^2 - m_b^2}{2 m_b} c^2}$$

## § Massless particle - photon

$E = |\mathbf{p}|c$ , massless particle only travels with velocity  $c$

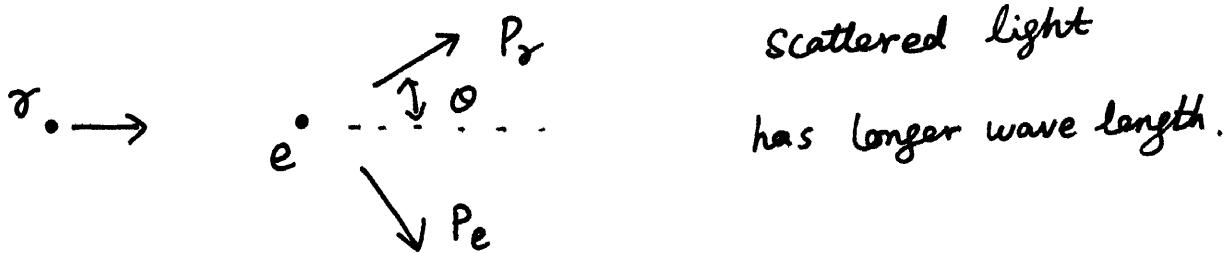
$$p^\mu p_\mu = \mathbf{p}^2 - \frac{E^2}{c^2} = 0$$

wave-interpretation  $\vec{p} = \hbar \vec{k}$ ,  $E = \hbar \omega$ ,  $\omega = kc$

$$p^\mu = \frac{\hbar \omega}{c} (\hat{k}, 1).$$

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§ Compton scattering: light scattered from free electrons  
inelastic



$$P_{\text{in},\gamma}^{\mu} = \frac{\hbar\omega_0}{c} (\hat{k}_0, 1) \quad P_{\gamma,\text{out}}^{\mu} = \frac{\hbar\omega}{c} (\hat{k}, 1)$$

$$P_{e,\text{in}}^{\mu} = (0, 0, 0, mc) \quad P_{e,\text{out}}^{\mu} = ?$$

$$P_{\text{in},\gamma}^{\mu} + P_{\text{in},e}^{\mu} = P_{\text{out},\gamma}^{\mu} + P_{e,\text{out}}^{\mu}$$

$$\Rightarrow P_{e,\text{out}}^{\mu} = P_{\text{in},\gamma}^{\mu} + P_{\text{int},e}^{\mu} - P_{\text{out},\gamma}^{\mu}$$

$$P_{e,\text{out}}^{\mu} \cdot P_{\mu,\text{out}}^{\mu} = P_{\text{in},\gamma}^{\mu} \cdot P_{\mu,\text{int}}^{\mu} + P_{\text{int},e}^{\mu} P_{\mu,\text{int}}^{\mu} + P_{\text{out},\gamma}^{\mu} P_{\mu,\text{out}}^{\mu}$$

$$+ 2 P_{\text{in},\gamma}^{\mu} P_{\mu,\text{in},e}^{\mu} - 2 P_{\text{in},\gamma}^{\mu} P_{\mu,\text{out},e}^{\mu} - 2 P_{\text{in},e}^{\mu} P_{\mu,\text{out}}^{\mu}$$

$$m_e^2 c^2 = m_e^2 c^2 + 0 + 0 + 2 P_{\text{in},e}^{\mu} (P_{\text{in},\gamma}^{\mu} - P_{\text{out},\gamma}^{\mu}) = P_{\text{in},\gamma}^{\mu} P_{\mu,\text{out}}^{\mu} = 0$$

$$\Rightarrow P_{\text{in},e}^{\mu} (P_{\text{in},\gamma}^{\mu} - P_{\text{out},\gamma}^{\mu}) = P_{\text{in},\gamma}^{\mu} P_{\mu,\text{out}}^{\mu}$$

only the 4th-component

$$mc \left[ \frac{\hbar\omega_0}{c} - \frac{\hbar\omega}{c} \right] = \frac{\hbar^2 \omega \omega_0}{c^2} [1 - \hat{k} \cdot \hat{k}_0]$$

is nonzero  $\Rightarrow mc(\omega_0 - \omega) = \frac{\hbar}{c} \omega \omega_0 (1 - \cos \theta)$

$$\Rightarrow \frac{1}{\omega} - \frac{1}{\omega_0} = \frac{\hbar}{mc^2} (1 - \cos \theta) \Rightarrow \lambda = \frac{c}{\omega} \Rightarrow \Delta \lambda = \frac{\hbar}{mc} (1 - \cos \theta)$$