

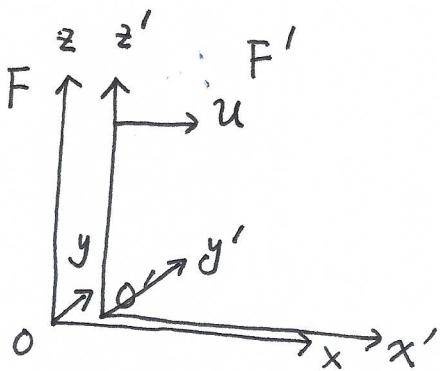
Lect 9 Electric fields of moving charges.

④ Recap of Lorentz transformations

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix}$$

$$\beta = \frac{u}{c}$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$



$$E'_x = E_x, \quad B'_x = B_x$$

$$\begin{pmatrix} E_y' \\ B_z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} E_y \\ B_z \end{pmatrix}, \quad \begin{pmatrix} E_z' \\ B_y' \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} E_z \\ B_y \end{pmatrix}$$

Consider the case that there is only electric field in F, then

$$E'_{||} = E_{||}, \quad E'_{\perp} = \gamma E_{\perp}, \quad \text{where } || \text{ and } \perp \text{ mean}$$

the field components along and perpendicular to the relative motion between F and F'.

⑤ 4-velocity and 4-momentum

$$X^\mu = (x, y, z, ct), \quad x_\mu = (x, y, z, -ct)$$

$\omega^\mu = \frac{dx^\mu}{d\tau}$ where τ is proper-time, which is a relativistic scalar.

Hence, ω^μ is a 4-vector

$$d\tau = dt \cdot \sqrt{1-\beta^2}$$

$$\Rightarrow \omega^\mu = \left(\frac{\vec{v}}{\sqrt{1 - v^2/c^2}}, \frac{c}{\sqrt{1 - v^2/c^2}} \right) \text{ is a 4-vector.}$$

It means that ω^μ satisfies the Lorentz transformation

$$\begin{pmatrix} \omega'^0 \\ \omega'^1 \\ \omega'^2 \\ \omega'^3 \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} \omega^0 \\ \omega^1 \\ \omega^2 \\ \omega^3 \end{pmatrix} \quad \omega'^1 = \omega^1 \text{ and } \omega'^2 = \omega^2$$

Actually, the above results are not easy to derive. I leave it as a homework problem.

Momentum $p^\mu = m_0 \omega^\mu = \left(\frac{m_0 \vec{v}}{\sqrt{1 - v^2/c^2}}, \frac{m_0 c}{\sqrt{1 - v^2/c^2}} \right)$

$$= (\vec{p}, E/c) \text{ is 4-momentum}$$

$$\boxed{\begin{pmatrix} p'^0 \\ p'^1 \\ p'^2 \\ p'^3 \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} p^0 \\ p^1 \\ p^2 \\ p^3 \end{pmatrix}, \quad p'^1 = p^1, \quad p'^2 = p^2, \quad p'^3 = p^3}$$

* 4-force.

Define $K^\mu = \frac{dp^\mu}{d\tau}$, $K^i = \frac{dp^i}{dt} \frac{dt}{d\tau} = \frac{F^i}{\sqrt{1 - v^2/c^2}}$

$$K^0 = \frac{dp^0}{dt} \frac{dt}{d\tau} = \frac{1}{c} \frac{dE}{dt} \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{\frac{1}{c} (\vec{F} \cdot \vec{U})}{\sqrt{1 - v^2/c^2}}$$

$$\begin{pmatrix} K' \\ K^0 \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} K' \\ K^0 \end{pmatrix} \Rightarrow \begin{aligned} K' &= \gamma(K' - \beta K^0) \\ K^0 &= \gamma(-\beta K' + K^0) \end{aligned}$$

In order to simplify, we prove the following identity

$$1 - \frac{v'^2}{c^2} = \left(1 - \frac{u^2}{c^2}\right) \left(1 - \frac{w^2}{c^2}\right) / \left(1 - \frac{v_x u}{c^2}\right)^2.$$

$$\text{Proof: } \frac{v'_x}{c} = \left(\frac{v_x}{c} - \frac{u}{c}\right) / \left(1 - \frac{v_x u}{c^2}\right), \quad \frac{v'_y}{c} = \frac{v_y/c}{1 - \frac{v_x u}{c^2}} \sqrt{1 - \frac{u^2}{c^2}}$$

$$\frac{v'_z}{c} = \frac{v_z/c}{1 - \frac{v_x u}{c^2}} \sqrt{1 - \left(\frac{u}{c}\right)^2}$$

$$\begin{aligned} 1 - \frac{v'^2}{c^2} &= 1 - \frac{1}{\left(1 - \frac{v_x u}{c^2}\right)^2} \left[\left(\frac{v_x}{c}\right)^2 + \left(\frac{u}{c}\right)^2 - 2\left(\frac{v_x u}{c^2}\right) + \left[1 - \left(\frac{u}{c}\right)^2\right] \left(\frac{v_y^2 + v_z^2}{c^2}\right) \right] \\ &= \frac{1}{\left(1 - \frac{v_x u}{c^2}\right)^2} \left[1 + \left(\frac{v_x u}{c}\right)^2 - \left(\frac{v_x}{c}\right)^2 - \left(\frac{u}{c}\right)^2 - \left[1 - \left(\frac{u}{c}\right)^2\right] \left(\frac{v_y^2 + v_z^2}{c^2}\right) \right] \\ &= \frac{1}{\left(1 - \frac{v_x u}{c^2}\right)^2} \left[1 - \left(\frac{u}{c}\right)^2 \right] \left[1 - \left(\frac{v}{c}\right)^2 \right] \end{aligned}$$

$$\text{or } \frac{1}{\sqrt{1 - \left(\frac{v'}{c}\right)^2}} = \gamma \frac{\left(1 - \frac{v_x u}{c}\right)}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}$$

$$\Rightarrow K' = \gamma(K' - \beta K^0) \rightarrow \frac{F'_x}{\sqrt{1 - \left(\frac{v'}{c}\right)^2}} = \gamma \left(\frac{F_x}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} - \frac{\beta \frac{1}{c} (\vec{F} \cdot \vec{U})}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} \right)$$

$$F'_x = \left(1 - \frac{v_x u}{c^2}\right)^{-1} \left[F_x - \frac{u_x}{c^2} (\vec{F} \cdot \vec{U}) \right]$$

$$K'_2 = K_2$$

$$\frac{F'_y}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} = \frac{F_y}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} \Rightarrow F'_y = \frac{F_y}{\gamma \left(1 - \frac{v_x u}{c^2}\right)}$$

Summary:

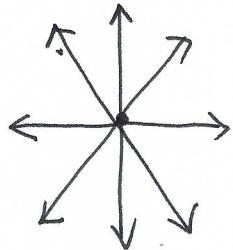
$$F'_x = \frac{F_x - \frac{u_x}{c} (\vec{F} \cdot \vec{v})}{1 - \frac{u_x u}{c^2}}$$

$$F'_y = \frac{\sqrt{1 - \left(\frac{u}{c}\right)^2} F_y}{1 - \frac{u_x u}{c^2}}, \quad F'_z = \frac{\sqrt{1 - \left(\frac{u}{c}\right)^2} F_z}{1 - \frac{u_x u}{c^2}}$$

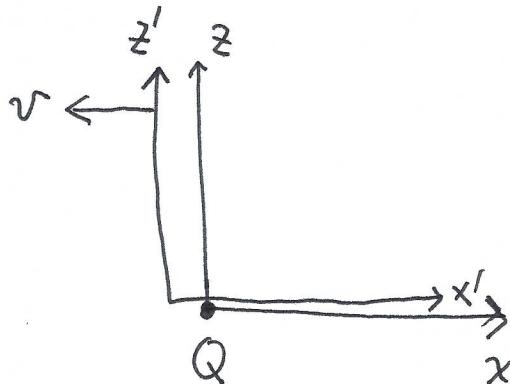
If, $\vec{v} = 0$, i.e. the velocity of a particle is zero, then

$$F'_x = F_x, \quad F'_y = \frac{1}{\gamma} F_y, \quad F'_z = \frac{1}{\gamma} F_z$$

{: Electric field of a moving charge at velocity v



Static



Q is static at
the origin of
the F-frame

$$E_x = \frac{Q x}{(x^2 + z^2)^{3/2}}$$

$$E_z = \frac{Q z}{(x^2 + z^2)^{3/2}}$$

Suppose Frame F' is moving along $-\hat{x}$ at the speed of v . \Rightarrow

$$\begin{pmatrix} x \\ t \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} x' \\ ct' \end{pmatrix}, \text{ then } Q \text{ is moving at } v \text{ along } \hat{x} \text{ in the frame of } F'.$$

At $\therefore Q$ passes the origin of F' , $t'=0$. At this moment of t'

$$\Rightarrow \begin{cases} x = \gamma x' \\ z = z' \end{cases}, \text{ the Fields measured in } F' \text{ should be}$$

$$E'_x(\vec{r}, t'=0) = E_x(\vec{r}, t) = \frac{Qx}{(x^2+z^2)^{3/2}} \\ = \frac{\gamma Qx'}{[(\gamma x')^2+z'^2]^{3/2}}$$

$$E'_z(\vec{r}, t'=0) = \gamma E_z(\vec{r}, t) = \frac{\gamma Qz}{(x^2+z^2)^{3/2}} = \frac{\gamma Qz'}{[(\gamma x')^2+z'^2]^{3/2}}$$

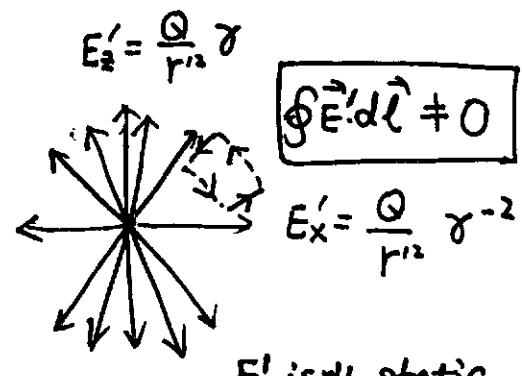
$$\Rightarrow \frac{E'_z(\vec{r}, t'=0)}{E'_x(\vec{r}, t'=0)} = \frac{z'}{x'} \Rightarrow$$

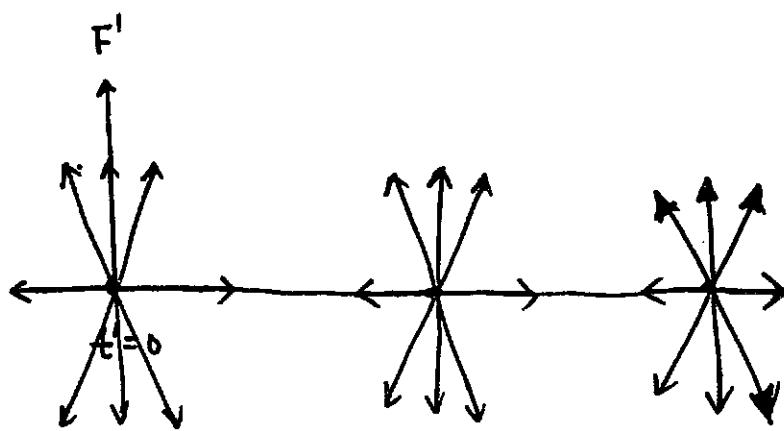
E' is along the radial direction
in F'

The total field

$$E^2 = E_x^2 + E_z^2 \Rightarrow E^2 = \frac{\gamma^2 Q^2 (x'^2 + z'^2)}{[(\gamma x')^2 + z'^2]^3} = \frac{Q^2 (x'^2 + z'^2)}{\gamma^4 [x'^2 + z'^2 (1 - \beta^2)]^3} \\ = \frac{Q^2 (1 - \beta^2)^2}{(x'^2 + z'^2)^2 (1 - \frac{\beta^2 z'^2}{x'^2 + z'^2})^3}$$

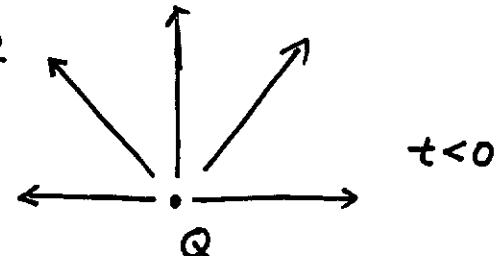
$$\Rightarrow E' = \frac{Q}{r'^2} \frac{1 - \beta^2}{(1 - \beta^2 \sin^2 \theta')^{3/2}}$$





§ Electric fields of a sudden moving charge

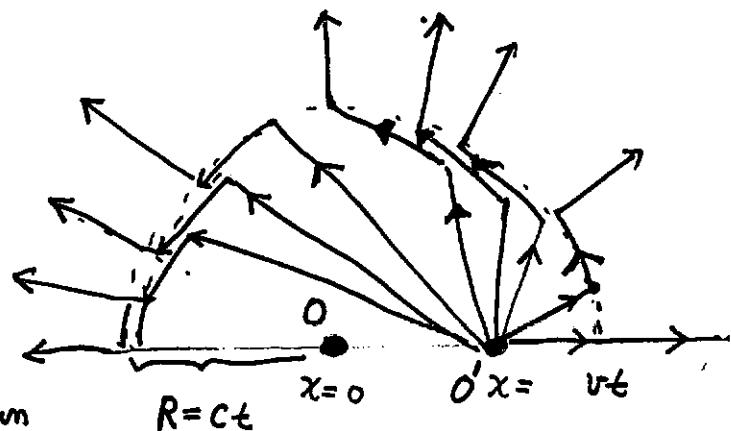
at $t < 0$, charge Q is at rest at the origin.



at $t > 0$, charge Q accelerates in a short time interval Δt , and then its velocity becomes v along \hat{x} direction, at $t > \Delta t$. Δt is a very short.

Then at the distance $R > ct$, the electric fields should not notice the motion of Q , thus should be the same as those at $t < 0$. At the distance $R < ct$, the field lines should be those of a moving charge at velocity v . The two different types of field lines should connect at a thin shell with the thickness ct .

Thus in the thin shell with the radius of $R = ct$, and the thickness of ct , the E fields are along the polar direction from the right pole to left pole.



These fields are transverse field, which are different from the electrostatic fields which are longitudinal.

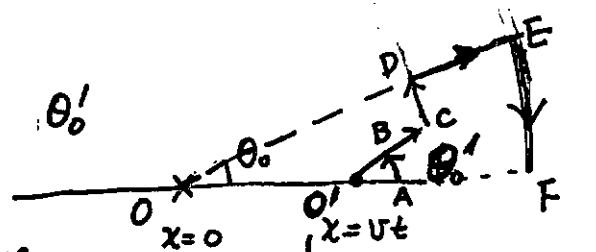
We need to decide how the two different regions are connected.

Consider the region of $A \rightarrow B \rightarrow C \rightarrow D \rightarrow E \rightarrow F$.

$A B$ is the area which spans the polar angle θ_0'

respect to O' . the fields on AB are of moving charge

$$\int_0^{\theta_0'} \sin\theta' d\theta' \cdot 2\pi Q \frac{1-\beta^2}{(1-\beta^2 \sin^2\theta')^{3/2}}$$



center of the moving charge

The flux passes the area span by $E \cdot F \Rightarrow \int_0^{\theta_0} \sin\theta d\theta \cdot 2\pi Q$
all other area don't contribute flux \Rightarrow

$$\int_0^{\theta_0'} \frac{\sin\theta' d\theta'}{(1-\beta^2 \sin^2\theta')^{3/2}} (1-\beta^2) = \int_0^{\theta_0} \sin\theta d\theta$$

$$\int \frac{\sin\theta d\theta'}{(1-\beta^2 \sin^2\theta')^{3/2}} = - \int \frac{d\cos\theta'}{(1-\beta^2 + \beta^2 \cos^2\theta')^{3/2}} = -\beta^{-3} \int \frac{d\cos\theta'}{(\cos^2\theta + \frac{1-\beta^2}{\beta^2})^{3/2}}$$

$$\int \frac{dx}{(a^2+x^2)^{3/2}} = \frac{x}{a^2(a^2+x^2)^{1/2}} + C$$

$$\Rightarrow \int_0^{\theta_0'} \frac{\sin\theta d\theta'}{(1-\beta^2 \sin^2\theta')^{3/2}} = -\beta^{-3} \left. \frac{\cos\theta'}{1-\beta^2 (\cos^2\theta + \frac{1-\beta^2}{\beta^2})^{1/2}} \right|_0^{\theta_0'}$$

$$= \beta^{-1} \left[\frac{\beta}{1-\beta^2} - \frac{\cos'\theta_0}{1-\beta^2 (\cos^2\theta_0 + \frac{1-\beta^2}{\beta^2})^{1/2}} \right]$$

$$\Rightarrow \beta^{-1} \left[\beta - \frac{\cos'\theta_0}{\beta^{-1} [1-\beta^2 \sin^2\theta_0]^{1/2}} \right] = [1-\cos\theta_0] \Rightarrow$$

$$\tan\theta_0' = \gamma \tan\theta_0$$

$$\cos\theta_0 = \frac{\cos\theta_0'}{\sqrt{1-\beta^2 \sin^2\theta_0}}$$

If we consider field lines like a rod, and every rod represents the same amount of flux, then the rods associated with the moving charge are steeper than those connecting to the rest position of the charge. Their relation is

$$\tan \theta' = \gamma \tan \theta.$$

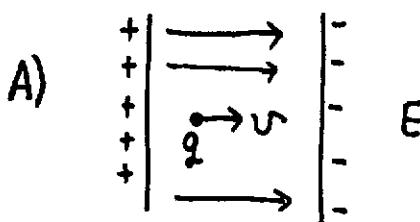
§ Forces on a moving charge without B field

we have assumed that for a static charge $\vec{F} = q\vec{E}$. We can actually show that for a moving charge, its electric force remains $\vec{F}_e = q\vec{E}$, (it may contain additional Lorentz force part which depends on velocity \vec{v}).

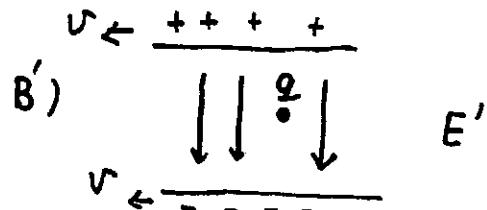
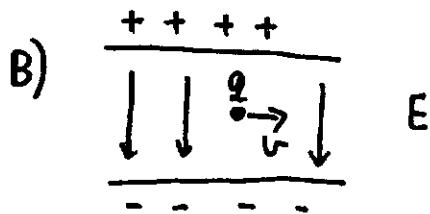
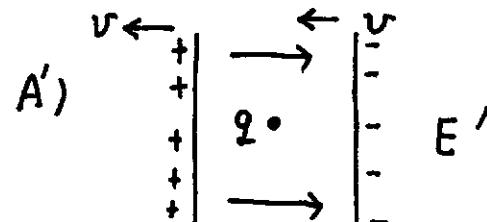
Let us consider two different frames: in the lab frame, particle is moving and electric field is static. In the particle's frame F' , particle is ~~moving~~, but electric field isn't.

Static

F (lab frame)

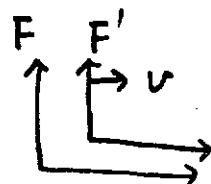


F' (particle frame)



(9)

$$\begin{pmatrix} P_x \\ E/C \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} P'_x \\ E'_C \end{pmatrix}$$



$$P_x > P'_x$$

$$\& \begin{pmatrix} x \\ ct \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} x' \\ ct' \end{pmatrix}$$

$$ct = \beta\gamma dx' + \gamma c dt' = \left(\beta\gamma \frac{dx'}{dt'} + \gamma c \right) dt'$$

$$\Rightarrow dt = (1 + \beta\gamma') \gamma dt' \quad \text{where } \beta' = \frac{1}{c} \frac{dx'}{dt'} = \frac{v}{c}$$

$$dP_x = \gamma dp'_x + \frac{\partial \gamma}{\partial C} dE' = \gamma (1 + \beta \frac{1}{c} \frac{dE'}{dp'_x}) dp'_x \quad \text{check } \frac{dE'}{dp'_x} = \frac{C^2 P'_x}{E'} = \beta'$$

$$\Rightarrow dP_x = \gamma (1 + \beta\gamma') dp'_x \quad \Rightarrow \frac{dP_x}{dt} = \frac{dp'_x}{dt'} \quad \& \frac{dp_y}{dt} = \frac{1}{\gamma} \frac{dp'_y}{dt'}$$

or: For two frames F & F', a particle is at rest in F', and F' moves at a velocity v respect to F. Decompose forces parallel and perpendicular to \vec{v} as $F_{||}, F'_{||}$ and F_{\perp}, F'_{\perp} , their relation

$$\boxed{\frac{dp_{||}}{dt} = \frac{dp'_{||}}{dt'} \quad \frac{dp_{\perp}}{dt} = \frac{1}{\gamma} \frac{dp'_{\perp}}{dt'}}$$

which is the same as E.

$$\text{then } \frac{dP'_{||}}{dt'} = q E'_{||} = q E_{||} = \frac{dp_{||}}{dt}$$

$$\frac{dp'_{\perp}}{dt'} = q E'_{\perp} = \gamma q E_{\perp} = \gamma \frac{dp_{\perp}}{dt} \quad \Rightarrow \quad \frac{dp_{||}}{dt} = q E_{||}, \quad \frac{dp_{\perp}}{dt} = q E_{\perp} \quad \checkmark$$

§ forces on a moving charge with B field

Let us consider a situation where \vec{E} is zero everywhere, the fast charge may still feel a velocity dependent force $\vec{F}_L(\vec{v})$. Generally speaking,

$F_{L,i} = T_{ij} v_j$, where T_{ij} should be rank-2 3-tensor. Moreover,

we expect that our system is an conservative system $\vec{F} \cdot \vec{v} = 0 \Rightarrow T_{ij} = -T_{ji}$.

This can be represented as a 3-axial vector

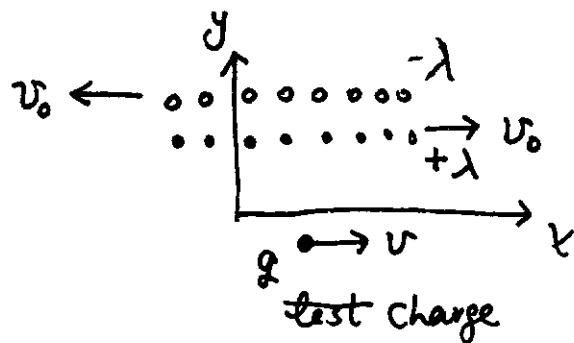
$$B_i = \frac{1}{2} \epsilon_{ijk} T_{jk} \Rightarrow \vec{F}_L = \frac{q}{c} \vec{v} \times \vec{B}.$$

only 3-independent
component.

The form of Lorentz force should be viewed as an experiment fact, rather than being derived. Nevertheless, we will give an explanation based on Lorentz transform

Lab frame: A line of positive charge moving at the speed of v_0 to the right
negative left.

In this frame, the line charge densities are $\pm \lambda$, respectively. Thus total charge is zero, no electric fields.



put a test charge moving at the speed of v to the right. What is the force on q ?

Let us change to the frame F' in which the test charge is at rest. Then the line charge densities $\lambda \pm$ are not equal any more due to different contraction. In this frame F' , the velocities of positive/negative charges are different

$$v'_+ = \frac{v_0 - v}{1 - \frac{v_0 v}{c^2}} \quad v'_- = \frac{v_0 + v}{1 + \frac{v_0 v}{c^2}} \quad \text{define } \beta'_\pm = \frac{v'_\pm}{c} \quad \beta_0 = \frac{v_0}{c}$$

$$\Rightarrow \boxed{\beta'_\pm = \frac{\beta_0 \mp \beta}{1 \mp \beta_0 \beta}}$$

For the positive charge, its line charge density in its rest frame should be $\lambda_{0,\pm} = \frac{\lambda}{\gamma_0}$: similarly $\lambda_{0,-} = -\frac{\lambda}{\gamma_0}$

$$\Rightarrow \text{in the frame } F' \Rightarrow \lambda'_\pm = \lambda_{0,\pm} \gamma'_\pm = \pm \frac{\lambda}{\gamma_0} \gamma'_\pm$$

$$\text{the net charge density } \Delta\lambda' = \lambda'_+ - \lambda'_- = \frac{\lambda}{\gamma_0} [\gamma'_+ - \gamma'_-]$$

$$\gamma'_+ - \gamma'_- = \frac{1}{\sqrt{1 - \left(\frac{\beta_0 - \beta}{1 - \beta\beta_0}\right)^2}} - \frac{1}{\sqrt{1 - \left(\frac{\beta_0 + \beta}{1 - \beta_0\beta}\right)^2}} = \frac{-2\beta_0\beta}{\sqrt{(1 - \beta_0^2)(1 - \beta^2)}} = -2\beta_0\beta\gamma_0\gamma$$

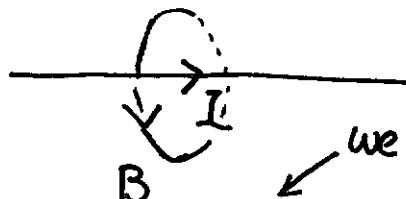
$$\Rightarrow \Delta\lambda' = -2\lambda\beta_0\beta\gamma \Rightarrow E'_y = \frac{2\Delta\lambda}{r} = -\frac{4\lambda\gamma v u_0}{rc^2}$$

$$F'_y = \frac{4\pi\lambda\gamma v u_0}{rc^2}$$

$$\text{in the Frame } F \Rightarrow F_y = \frac{1}{\gamma} F'_y = \frac{4\pi\lambda v u_0}{rc^2} = \boxed{\frac{2I}{rc}} \frac{q v}{c}$$

$$\text{where } I = 2\lambda u_0$$

$\text{Ampere's law } B = \frac{2I}{rc}$



we don't have magnetic-monopole.

Magnetic field from electric charge is indeed an relativistic effect. Because electric force usually are cancelled due to charge neutrality, magnetic force can appear! E & M are naturally relativistic, although people didn't realize it until Einstein pointed it out!