Problem 1. Vector algebra

1) Prove the following relations.

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$$
$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$$
$$(\mathbf{A}\times\mathbf{B})\times\mathbf{C} = -\mathbf{A}(\mathbf{B}\cdot\mathbf{C}) + \mathbf{B}(\mathbf{A}\cdot\mathbf{C})$$
$$(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) = (\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D}) - (\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C})$$

2) What are the differences and similarities between a 3-dimensional vector and a pseudo-vector under spatial transformations? Give a few examples of pseudo-vectors.

Problem 2. The vector calculus
Prove the following identities of vector calculus.

1) $f(\mathbf{r})$ and $g(\mathbf{r})$ are two scalar functions. Prove that

$$\begin{aligned} \nabla(fg) &= f\nabla g + (\nabla f)g \\ \nabla(f/g) &= \tfrac{1}{g}\nabla f - f(\nabla g)/g^2 \end{aligned} \tag{1}$$

2) $\mathbf{A}(\mathbf{r})$ and $\mathbf{B}(\mathbf{r})$ are vector field functions. Prove that

$$\begin{aligned} \nabla \cdot (f\mathbf{A}) &= f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f) \\ \nabla \times (f\mathbf{A}) &= f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f) \\ \nabla \cdot (\mathbf{A} \times \mathbf{B}) &= \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}). \end{aligned} \tag{2}$$

3) This part is not required, and I list a few formulas for future convenience. If you can prove some, you will get extra credits. They are significantly more complicated. If you are interested, TA will teach you some tricks.

$$\begin{aligned} \nabla \times (\mathbf{A} \times \mathbf{B}) &= (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) \\ \nabla(\mathbf{A} \cdot \mathbf{B}) &= (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) \\ \nabla \times (\nabla \times \mathbf{A}) &= \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \end{aligned} \tag{3}$$

Problem 3. Fundamental theorems of vector calculus

The electric field from a point charge q is

$$\mathbf{E}(\mathbf{r}) = \tfrac{q}{r^3}\mathbf{r}. \tag{4}$$

1) Show that $\nabla \cdot \mathbf{E}(\mathbf{r}) = 0$ at $\mathbf{r} \neq 0$. Calculate the surface integral of the electric flux

$$\oint \mathbf{E} \cdot d\mathbf{s} = ? \tag{5}$$

Does this result violate the Gauss's theorem?

3) Prove that

$$\nabla \times \mathbf{E} = 0, \tag{6}$$

and

$$\oint d\mathbf{r} \cdot \mathbf{E} = 0, \tag{7}$$

for an arbitrary loop. Prove that this means that $\mathbf{E}$ can be represented as a gradient of a scalar function $\phi(\mathbf{r})$. Derive this function which is called the electric potential.

Problem 4. Multi-valued functions

1) In a superconductor, each position $\mathbf{r}$ is associated with a complex number $\Delta(\mathbf{r}) = |\Delta(\mathbf{r})|e^{i\theta(\mathbf{r})}$. Prove that for an angle distribution function $\theta(\mathbf{r})$,

$$\oint d\mathbf{r} \cdot \nabla\theta(\mathbf{r}) = 2n\pi, \tag{8}$$

where n is an integer. Search in the literature to find the physical meaning of n.

2) Based on the Stokes theorem, derive that

$$\nabla \times \nabla\theta(\mathbf{r}) = ? \tag{9}$$