Problem 1. Partial integral.

1) Prove

$$\iiint dv f(\nabla \cdot \mathbf{A}) = \oint f\mathbf{A} \cdot d\mathbf{a} - \iiint dv \mathbf{A} \cdot \nabla f, \tag{1}$$

where $\iiint dv$ means a volume integral, and $\oint d\mathbf{a}$ is the integral over the surface surrounding this volume. f is a scalar function, and $\mathbf{A}$ is a vector function.

2) We extend the volume integral to the entire space. In many physical applications, $f(\nabla \cdot \mathbf{A})$ decays faster than $1/r^2$ as $r \to \infty$. Simplify the above result in this case.

Problem 2. Spherical and cylindrical coordinates
1) The unit vectors of the spherical coordinates are denoted as $(\hat{\mathbf{e}}_{\mathbf{r}}, \hat{\mathbf{e}}_{\theta}, \hat{\mathbf{e}}_{\phi})$. Please work out the relation between

$$(\hat{\mathbf{e}}_{\mathbf{r}}, \hat{\mathbf{e}}_{\theta}, \hat{\mathbf{e}}_{\phi}) = (\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}})O, \tag{2}$$

where O is a 3×3 orthogonal matrix. Write the explicit form of O.

2) Work out the relation

$$(d\hat{\mathbf{e}}_{\mathbf{r}}, d\hat{\mathbf{e}}_{\theta}, d\hat{\mathbf{e}}_{\phi}) = (\hat{\mathbf{e}}_{\mathbf{r}}, \hat{\mathbf{e}}_{\theta}, \hat{\mathbf{e}}_{\phi})T, \tag{3}$$

where T is a 3×3 antisymmetric matrix. Write the explicit form of T.

3) Now we change to the cylindrical coordinates. Work out the relation between the unit vectors of the cylindrical coordinates and those of the Cartesian coordinates.

$$(\hat{\mathbf{e}}_{\rho}, \hat{\mathbf{e}}_{\varphi}, \hat{\mathbf{e}}_{\mathbf{z}}) = (\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}})O'. \tag{4}$$

Work out the relation

$$(d\hat{\mathbf{e}}_{\rho}, d\hat{\mathbf{e}}_{\varphi}, d\hat{\mathbf{e}}_{\mathbf{z}}) = (\hat{\mathbf{e}}_{\rho}, \hat{\mathbf{e}}_{\varphi}, \hat{\mathbf{e}}_{\mathbf{z}})T'. \tag{5}$$

Write down the explicit forms of O' and T'.

Problem 3. Vector calculus in the curvilinear coordinates

Under the spherical coordinate, $d\mathbf{r} = dr\hat{\mathbf{e}}_{\mathbf{r}} + rd\theta\hat{\mathbf{e}}_\theta + r\sin\theta d\phi\hat{\mathbf{e}}_\phi$. Under the cylindrical coordinate, $d\mathbf{r} = d\rho\hat{\mathbf{e}}_\rho + rd\varphi\hat{\mathbf{e}}_\varphi + dz\hat{\mathbf{z}}$.

The formula of the gradient for a scalar function $f(u^1, u^2, u^3)$ in orthogonal curvilinear coordinates is

$$\nabla f = \sum_i \frac{1}{\sqrt{g_{ii}}} \frac{\partial}{\partial u^i} f. \tag{6}$$

For a vector field $\mathbf{v} = v^1\hat{\mathbf{e}}_{\mathbf{u^1}} + v^2\hat{\mathbf{e}}_{\mathbf{u^2}} + v^3\hat{\mathbf{e}}_{\mathbf{u^3}}$, its divergence is

$$\nabla \cdot \mathbf{v} = \frac{1}{\sqrt{|detg|}} \sum_i \frac{\partial}{\partial u^i} \sqrt{|\frac{detg}{g_{ii}}|} v^i \tag{7}$$

(The Einstein notation is not assumed here.)

1) Write down the metric matrix g for the spherical and cylindrical coordinates, such that $(dr)^2 = g_{ij}du^i du^j$, with $du^1 = dr$, $du^2 = d\theta$, $du^3 = d\phi$.

2) Write down the volume element dv in terms of the spherical and cylindrical coordinates.

3) Work out the expression of the Laplacian ∇^2 operator in terms of the spherical coordinates and the cylindrical coordinates.

Problem 4. Levi-civita symbol Prove the following identity

$$\epsilon_{ijk}\epsilon_{ij'k'} = \delta_{jj'}\delta_{kk'} - \delta_{jk'}\delta_{kj'}, \tag{8}$$

where the repeated symbol represents summation.