Problem 1: The 4-momentum

By using the relativistic velocity addition law, please directly prove that the 4-momentum defined as $(\mathbf{p}, p^0)$

$$\mathbf{p} = \frac{m_0 \mathbf{v}}{\sqrt{1-(v/c)^2}}, \quad \mathbf{p}^0 = \frac{m_0 c}{\sqrt{1-(v/c)^2}}, \tag{1}$$

satisfies the Lorentz transformation

$$\begin{pmatrix} p'^{,1} \\ p'^{,0} \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} p^1 \\ p^0 \end{pmatrix}. \tag{2}$$

We suppose that the frame F' moves along the x-direction at the velocity u relative the the frame of F. $\beta = u/c$ and $\gamma = 1/(1-\beta^2)$. The quantities with a prime are the ones measured in the frame F', and those without a prime are measured in the frame F.

You may need to use the following relation.

$$1 - \frac{v'^{,2}}{c^2} = \frac{\left(1-\frac{u^2}{c^2}\right)\left(1-\frac{v^2}{c^2}\right)}{\left(1-\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\right)^2} \tag{3}$$

You are also encouraged to prove this.

Problem 2: Wavevector and frequency

Prove that the wavevector $\mathbf{k}$ and frequency ω of a propagating wave combined in the way of

$$k^\mu = (\mathbf{k}, k^0 = \tfrac{\omega}{c}), \tag{4}$$

satisfy 4-vector according to the Lorentz transformation.

Please prove it in two different ways.

1) You may use the fact that the phase difference over a space-time interval $(\mathbf{\Delta x}, \Delta t)$, i.e, $\mathbf{k} \cdot \mathbf{\Delta x} - \omega \Delta t$ is a Lorentz invariant to prove the above result.

2) Consider a special case that $\mathbf{k}$ is along the x-direction, and the relative motion between F' and F frame is also along x-direction. You may examine how to define wave-lengths and periods in these two different frames.

Problem 3: Space-time derivatives
Consider the Lorentz transform between two frames F and F' in the case that F' is moving at the velocity u along the x-direction with respect to F, i.e.,

$$\begin{pmatrix} x'^{,1} \\ x'^{,0} \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} x^1 \\ x^0 \end{pmatrix}, \tag{5}$$

where $\beta = u/c$ and $\gamma = 1/(1-\beta^2)$.

Prove that the derivatives defined in the following way $\partial^1 = \frac{\partial}{\partial x}$, and $\partial^0 = -\frac{1}{c}\frac{\partial}{\partial t}$ also satisfy the above Lorentz transforamtion.

More problems

Work out the following problems in Berkeley Physics Course Volume I,
Problem 14 of Chapter 11,
all the problems of the advanced topics of Chapter 12,
Problem 6 of Chapter 13,
Problem 13 of Chapter 13 (You need to read the text on the Inverse Compton scattering).