1 Relativistic invariance

Prove that $\mathbf{E}\cdot\mathbf{B}$ and $\mathbf{E}^2-\mathbf{B}^2$ are invariant under the Lorentz transformation of the electromagnetic field.

2. and 3. **Berkeley Vol II Problems 9.5, 9.7**

4 Angular momentum of a charge-monopole system

We learned that angular momentum conservation is a consequence of spatial isotropy. Below we examine a subtle case, an isotropic system but not a central-force one.

Consider a magnetic monopole fixed at the origin with a magnetic charge g. Similar to an electric charge, the magnetic field generated is $\vec{B} = g\vec{r}/r^3$. Below we study the motion of a particle with an electric charge q and mass m in this monopole magnetic field. You may use either the Gaussian unit or the SI unit for the Lorentz force.

1) Prove that the kinetic energy of this particle is conserved, or, equivalently, its speed v of motion is a constant.

2) The mechanical angular momentum is defined as usual $\vec{L}' = m\vec{r} \times \vec{v}$. Please show that it actually is NOT conserved, i.e., $\frac{d}{dt}\vec{L}' \neq 0$.

3) Since angular momentum conservation is a requirement of spatial isotropy, we should check why the above naive definition does not work. The answer is that there is an extra contribution $\vec{L}_{EM}$ not taken into account yet, which arises from the angular momentum of the E&M field. Then the total angular momentum is defined as $\vec{L} = \vec{L}' + \vec{L}_{EM}$.

Based on symmetry analysis, please prove that $\vec{L}_{EM}$ is along the radial direction (No actual calculation is needed for this statement.). Then we express $\vec{L}_{EM} = k\hat{r}$, where k is a constant to be determined and $\hat{r}$ is the unit vector along the radial direction.

We need to carefully chose the expression of k such that $\frac{d}{dt}\vec{L} = 0$. In order for this purpose, what is k?

4) In quantum mechanics, a non-zero angular momentum has a minimal quantized value. Apply this quantization to the charge-monopole system, what will you obtain from it?

5 (E&M) The Valentine's day monopole.

The existence of the magnetic monopole cannot be ruled out by any general principle. We need to consider how to detect its existence and how to revise the Maxwell equations to include them.

The experimental detection of magnetic monopole so far has been inconclusive. Nevertheless, there was a Valentine's day monopole event at Stanford on Feb. 14, 1982.

We assume in the static case that the magnetic version of Coulomb's law is $\vec{B} = \frac{g}{r^2}\hat{r}$ (Gauss unit), or $\vec{B} = \frac{\mu_0}{4\pi}\frac{g}{r^2}\hat{r}$ (SI unit), for a point magnetic charge g.

1) What is the Gauss's law for magnetic monopoles in the static case, i.e. $\nabla \cdot \vec{B}(\vec{r}) =$? Use the $\rho_m(\vec{r})$ to denote the monopole density.

2) The detector at Stanford was basically a superconducting coil with a self-inductance L, and an electric current of I flowing around it without dissipation. Imagining that a magnetic monopole with a magnetic charge g from a distant place penetrates through the coil and flies away, what do you expect on how I changes? In comparison, if a magnetic dipole, say, a permanent magnet, penetrates this coil, how different will the phenomenon be?

That signal you would expect indeed showed up on Feb 14, 1982, but could never be repeated later. Maybe it means that a monopole came once but never returned, or, it was just a mistake.

3) Suppose that monopole charges are also conserved as electric charges do. Write down the expression of the monopole charge conservation in terms of ρ_m and $\vec{j}_m$, where $\vec{j}_m(\vec{r})$ is the monopole current density.

4) Prove that the previous Maxwell equation of Faraday's law does not satisfy the monopole charge conservation. In order to maintain the monopole charge conservation, how should Faraday's law be modified?

5) Imagine an infinitely long wire carrying a steady monopole current I_m along the $\hat{z}$-direction, which is the spatial distribution of the electric field $\vec{E}(\rho, \phi, z)$ using the cylindrical coordinates?