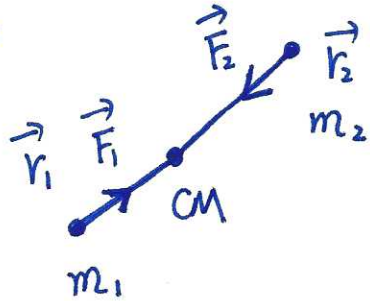
Lect 12 Kepler problem (I)

- CM and relative coordinates : reduced mass



$$\begin{cases} \vec{F}_1(|\vec{r}_1-\vec{r}_2|) = -\vec{F}_2(|\vec{r}_1-\vec{r}_2|) & \\ m_1\ddot{\vec{r}}_1 = \vec{F}_1 & (1) \\ m_2\ddot{\vec{r}}_2 = \vec{F}_2 & (2) \end{cases}$$

$(1)+(2)=0 \Rightarrow \boxed{\ddot{\vec{R}}=0}$ with $\boxed{\vec{R}=\frac{m_1\vec{r}_1+m_2\vec{r}_2}{m_1+m_2}}$ Center of mass coordinate

$\frac{(1)}{m_1}-\frac{(2)}{m_2} \Rightarrow \ddot{\vec{r}} = \left(\frac{1}{m_1}+\frac{1}{m_2}\right)\vec{F}_1$ where $\boxed{\vec{r}=\vec{r}_1-\vec{r}_2}$ relative coordinate

$$\boxed{\mu\ddot{\vec{r}} = \vec{F}_1(|r|)}$$

$\mu = \frac{m_1m_2}{m_1+m_2}$ ← reduced mass.

$\mu < m_1, m_2$

- seperation of center of mass motion and relative motion
- For the relative motion, it's reduced to a single mass point moving in a centeral force field $\vec{F}_1(|r|)$. The mass is replaced by μ.
- $T = \frac{1}{2}m_1\dot{\vec{r}}_1^2 + \frac{1}{2}m_2\dot{\vec{r}}_2^2$ plug in $\begin{cases} \vec{r}_1 = \vec{R} + \frac{m_2}{M}\vec{r} \\ \vec{r}_2 = \vec{R} - \frac{m_1}{M}\vec{r} \end{cases}$ with $M=m_1+m_2$

$$= \frac{1}{2}m_1\left[\dot{\vec{R}}^2 + \left(\frac{m_2}{M}\right)^2\dot{\vec{r}}^2 + 2\dot{\vec{R}}\cdot\dot{\vec{r}}\frac{m_2}{M}\right]$$

$$+ \frac{1}{2}m_2\left[\dot{\vec{R}}^2 + \left(\frac{m_1}{M}\right)^2\dot{\vec{r}}^2 - 2\dot{\vec{R}}\cdot\dot{\vec{r}}\frac{m_1}{M}\right] = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2$$

- $E = T + \mathcal{U} = \frac{1}{2} M \dot{\vec{R}}^2 + \underbrace{\frac{1}{2} \mu \dot{\vec{r}}^2 + \mathcal{U}(r)}_{\text{relative motion}}$

- $\vec{L}_{CM}$ in the CM frame, i.e, the frame that $\vec{R}$ is at rest.

$$\vec{L}_{CM} = (\vec{r}_1 - \vec{R}) \times m_1 (\dot{\vec{r}}_1 - \dot{\vec{R}}) + (\vec{r}_2 - \vec{R}) \times m_2 (\dot{\vec{r}}_2 - \dot{\vec{R}})$$

$$= \frac{m_2}{M} \vec{r} \times m_1 \frac{m_2}{M} \dot{\vec{r}} + \left(-\frac{m_1}{M} \vec{r}\right) \times m_2 \left(-\frac{m_1}{M}\right) \dot{\vec{r}}$$

$$= \frac{m_1 m_2}{M} \left(\frac{m_2 + m_1}{M}\right) \vec{r} \times \dot{\vec{r}} = \boxed{\mu \vec{r} \times \dot{\vec{r}} = \vec{L}_{CM}}$$

- Reduction to 1D motion

We have reduced the 2-body problem into a single body problem in 3D. Now let us further reduce it to 2D and to 1D motion. In the CM frame, $\vec{L}_{CM}$ is conserved!

The force passes the origin $\longrightarrow$ no torque.

(Angular momentum conservation due to spatial isotropy).

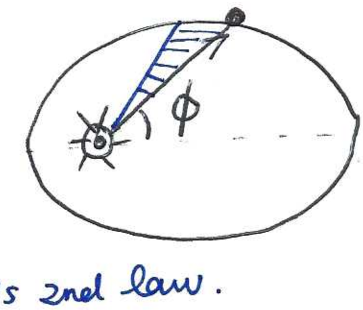
$$\frac{d}{dt} \vec{L}_{CM} = 0 \quad \Rightarrow \quad \vec{L}_{CM} \equiv \text{const vector}.$$

$\vec{L}_{cm}$ is perpendicular to the orbiatal plane $\Rightarrow$

$\boxed{\text{the motion is co-planar, say, in the xy-plane, and } \vec{L}_{cm} = l\hat{z}.}$

Then we use the equation of motion in the polar system

$$\begin{cases} F_r = \mu(\ddot{r} - r\dot{\phi}^2) & \text{①} \\ F_\phi = \mu(r\ddot{\phi} + 2\dot{r}\dot{\phi}) = \frac{1}{r}\mu\frac{d}{dt}(r^2\dot{\phi}) & \text{②} \end{cases}$$



$F_\phi = 0 \Rightarrow \frac{d}{dt}[\mu r^2\dot{\phi}] = 0$ ← This is Kepler's 2nd law.

Actually $\vec{L}_{cm} = l\hat{z} = \mu r\hat{r} \times \vec{v} = \mu r\hat{r} \times [\dot{r}\hat{r} + r\frac{d\hat{r}}{d\phi}\dot{\phi}]$

$$= \mu r^2\dot{\phi}[\hat{r} \times \hat{\phi}] = \mu r^2\dot{\phi}\hat{z}$$

$\Rightarrow \mu r^2\dot{\phi} = l \Rightarrow \dot{\phi} = \frac{l}{\mu r^2} \Rightarrow r\dot{\phi}^2 = \frac{l^2}{\mu^2 r^3}$

$\Rightarrow F_r = \mu\ddot{r} - \frac{l^2}{\mu r^3} \Rightarrow \boxed{\mu\ddot{r} = F_r + \frac{l^2}{\mu r^3}}$ ← effective 1D motion

Similarly, we can apply our previous knowledge on 1D motion to reduce it to 1st differential Eq.

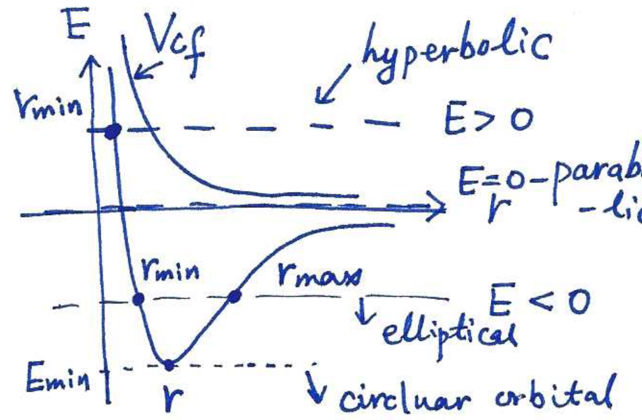
$$E = \frac{1}{2}\mu\dot{r}^2 + \underbrace{U(r) + \frac{l^2}{2\mu r^2}}$$ where $U(r) = -\int_{r_0}^{r} F_r\,dr$

$$\boxed{E = \frac{1}{2}\mu\dot{r}^2 + U_{eff}(r)}$$

The effect of angular mometum is included by $\frac{l^2}{2\mu r^2} \triangleq V_{cf}(r)$

For Kepler problem $U(r) = -\frac{Gm_1m_2}{r} = -\frac{\gamma}{r}$ (where $\gamma = Gm_1m_2$)

$U_{eff}(r) = -\frac{\gamma}{r} + \frac{\ell^2}{2\mu r^2}$



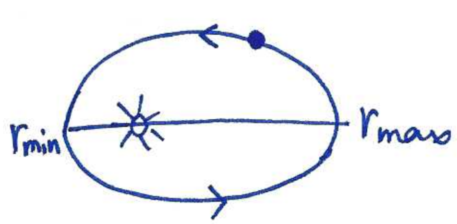
① E < 0 : bound orbital

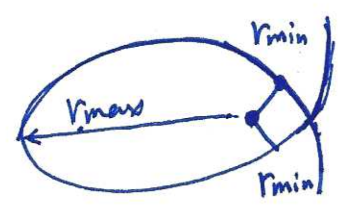
at E_{min}, the radial motion is at rest → circular motion

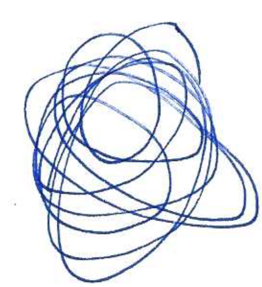
② E = 0 and E > 0 unbounded orbitals

What's special of $1/r^2$ - force field? — closed orbital at E < 0

① The period of radial motion (bounce) is the same as angular period ϕ from $0 \sim 360°$.







② for general central force, the orbit may not be closed!

The ellipse may process. The angular period isn't the same as the radial period.

- Solve the equation of orbit

$$\begin{cases} \mu \ddot{r} = F_r + \dfrac{\ell^2}{\mu r^3} & ① \longrightarrow \text{Solve } r(\phi) \\ \dot{\phi} = \dfrac{\ell}{\mu r^2} & ② \end{cases}$$

define $u = 1/r$ and we replace $\frac{d}{dt}$ by $\frac{d}{d\phi}$

$$\frac{d}{dt} = \frac{d\phi}{dt}\frac{d}{d\phi} = \frac{\ell}{\mu r^2}\frac{d}{d\phi} = \frac{\ell u^2}{\mu}\frac{d}{d\phi}$$

$$\dot{r} = \frac{\ell u^2}{\mu}\frac{d}{d\phi}\left(\frac{1}{u}\right) = -\frac{\ell}{\mu}\frac{du}{d\phi}$$

$$\ddot{r} = -\frac{\ell}{\mu}\frac{d}{dt}\frac{du}{d\phi} = -\frac{\ell}{\mu}\frac{\ell u^2}{\mu}\frac{d^2u}{d\phi^2} \Rightarrow -\frac{\ell^2 u^2}{\mu^2}\frac{d^2u}{d\phi^2} = \frac{1}{\mu}F_r + \frac{\ell^2}{\mu^2}u^3$$

or $\boxed{\dfrac{d^2u}{d\phi^2} = -u(\phi) - \dfrac{\mu}{\ell^2 u^2}F_r}$ plug in $F_r = -\dfrac{\gamma}{r^2} = -\gamma u^2$

$$\Rightarrow \frac{d^2u}{d\phi^2} = -u + \frac{\mu\gamma}{\ell^2} \leftarrow \text{inhomgenous 2nd order linear differential Eq}$$

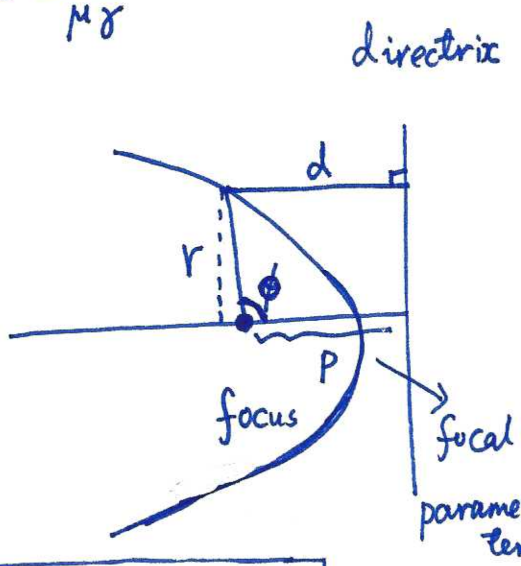
$$u = \underbrace{A\cos(\phi - \delta)}_{\text{solution to the homogenous part}} + \frac{\mu\gamma}{\ell^2} \leftarrow \text{a special solution}$$

δ can be choose by choosing the x-axis along the angle δ-direction i.e. major axis.

$$\Rightarrow \frac{1}{r} = \frac{\mu\gamma}{\ell^2}\left[1 + e\cos\phi\right], \quad \text{where } e = \frac{A\ell^2}{\mu\gamma}$$

$$\Rightarrow \quad r(\phi) = \frac{c}{1 + e\cos\phi} \qquad \text{with} \quad \begin{cases} c = \dfrac{l^2}{\mu\gamma} \\ e = \dfrac{Al^2}{\mu\gamma} \end{cases}$$

§ conic curves / sections

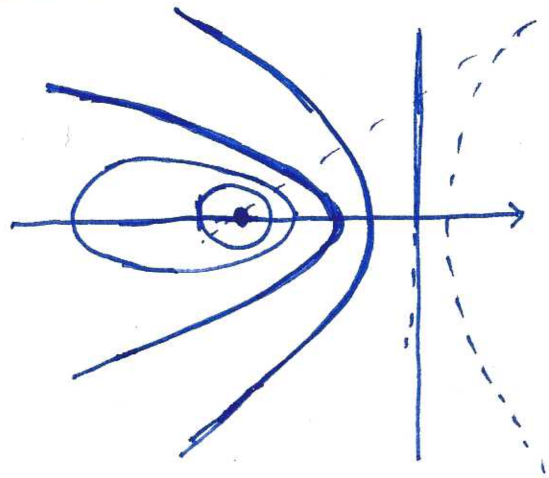


p: focal parameter

e: eccentricity

$$e = \frac{r}{d} \quad \text{with} \quad d = p - r\cos\theta$$

$$\Rightarrow \quad ed = ep - er\cos\theta = r \quad \Rightarrow \quad \boxed{r = \frac{ep}{1 + e\cos\phi}}$$



$0 < e < 1$ — ellipse

$e = 1$ — parabola

$e > 1$ — hyperbola

change to Cartisican coordinate

$$r = ep - er\cos\phi \quad \leftarrow \quad r\cos\phi = x$$

$$x^2 + y^2 = (ep)^2 + e^2x^2 - 2e^2p\,x$$

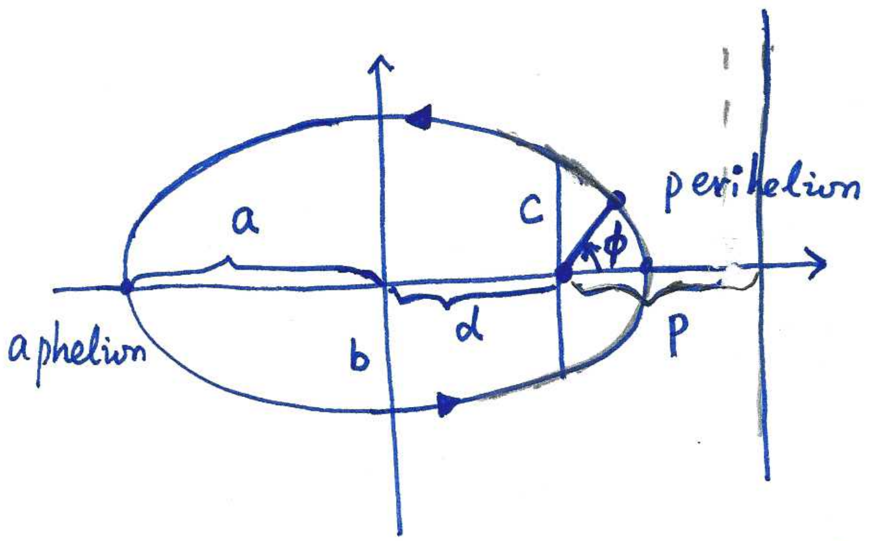
$$\boxed{(1-e^2)\left[x + \frac{e^2p}{1-e^2}\right]^2 + y^2 = \frac{e^2p^2}{1-e^2}}$$

for $0<e<1 \Rightarrow \dfrac{\left(x+\frac{e^2p}{1-e^2}\right)^2}{\left(\frac{ep}{1-e^2}\right)^2} + \dfrac{y^2}{\left(\frac{ep}{\sqrt{1-e^2}}\right)^2} = 1$

$$\Rightarrow \begin{cases} a = \dfrac{c}{1-e^2} \\ b = \dfrac{c}{\sqrt{1-e^2}} \end{cases}$$

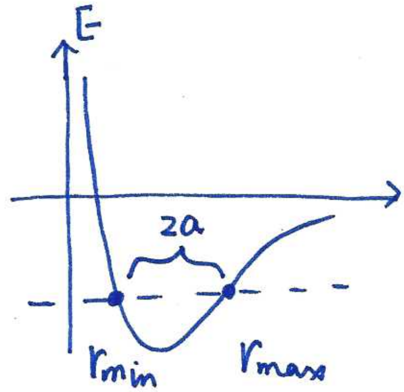
$$\begin{cases} c = ep = \dfrac{\ell^2}{\mu\gamma} \\ d = \dfrac{e^2p}{1-e^2} = ea \end{cases}$$

$$\begin{cases} e = \dfrac{A\ell^2}{\mu\gamma} \\ p = \dfrac{1}{A} \end{cases}$$



§ Express the orbit by using conserved quantities

• Energy: using the effective potential



$$U_{eff}(r) = -\frac{\gamma}{r} + \frac{\ell^2}{2\mu r^2}$$

$$r_{min} = \frac{c}{1+e} = \frac{\ell^2}{\mu\gamma(1+e)}$$

$$E = -\frac{\gamma}{r_{min}} + \frac{\ell^2}{2\mu r_{min}^2} = \frac{1}{2r_{min}}\left[\frac{\ell^2}{\mu r_{min}} - 2\gamma\right] = \left(\frac{\ell^2}{2\mu\gamma}\right)^{-1}(1+e)\gamma(e-1)$$

$$= \frac{\gamma^2\mu}{2\ell^2}(e^2-1) = \frac{-\gamma}{2a}$$

$$\boxed{a = \frac{\gamma}{-2E}}$$

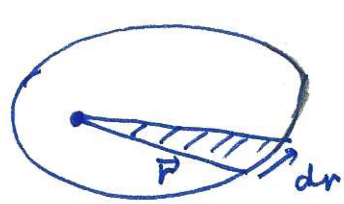
- the half-major axis "a" is only determined by the energy.
- The half Latus-rectum (Cord length) $\boxed{c = \frac{\ell^2}{\mu\gamma}}$ is only dertmined by the angular momentum

- $a = \frac{c}{1-e^2} \Rightarrow 1-e^2 = \frac{c}{a} = \frac{\ell^2}{\mu\gamma}\cdot\frac{-2E}{\gamma} \Rightarrow e = \sqrt{1+\frac{2\ell^2 E}{\mu\gamma^2}}$

$$\frac{b^2}{a^2} = 1-e^2 \Rightarrow \frac{b^2}{a} = (1-e^2)\,a = c \Rightarrow b = \sqrt{\frac{\ell^2}{-2\mu E}}$$

§ Kepler's 3rd law

$$d\vec{A} = \frac{1}{2}\vec{r}\times d\vec{r} \Rightarrow$$



$$\frac{dA}{dt} = \frac{1}{2}\frac{\ell}{\mu}$$

The total area $A = \underbrace{}_{\pi ab} \Rightarrow \tau = \frac{A}{dA/dt} = \frac{2\pi ab\mu}{\ell}$

$$\Rightarrow \tau^2 = \frac{4\pi^2 a^2 a^2 (1-e^2)\mu^2}{\ell^2} = \frac{4\pi^2 a^3 c\mu^2}{\ell^2} = \frac{4\pi^2 a^3 \mu}{\gamma}$$

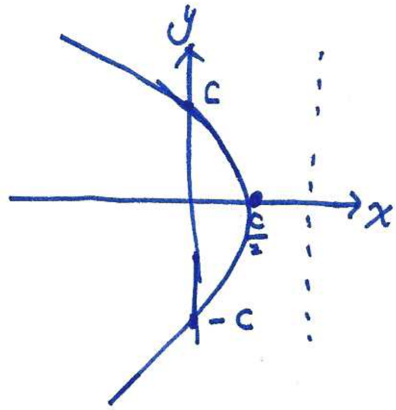
plug in $c = \frac{\ell^2}{\mu\gamma}$

$$\Rightarrow \boxed{\frac{\tau^2}{a^3} = \frac{4\pi^2\mu}{\gamma} = \frac{4\pi^2}{G M_{sun}}}$$

$$\gamma = G m_1 m_2 = G\mu(m_{sun} + M_{earth}) \approx G\mu M_{sun}$$

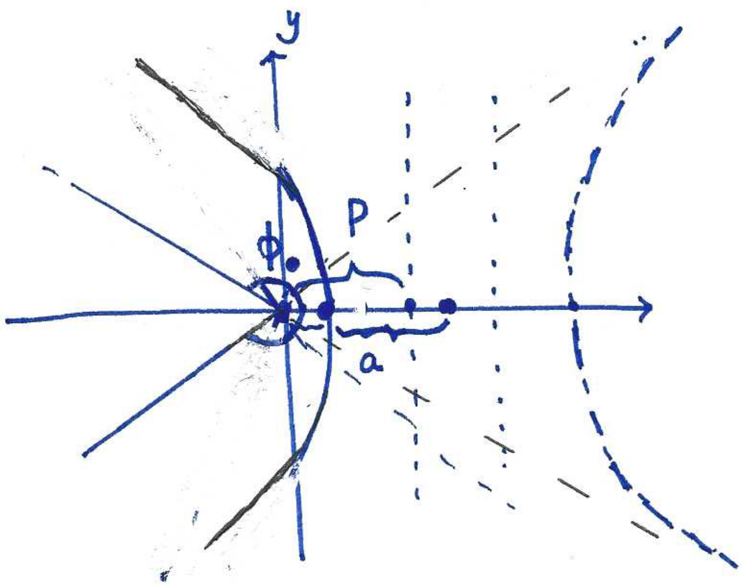
Kepler problem (II)

§ unbounded orbits : $r(\phi) = \dfrac{c}{1 + e\cos\phi}$



① $e = 1 \Rightarrow r(\phi = \pi) \to +\infty$, $y^2 = -2c\left[x - \frac{c}{2}\right]$

② $e > 1$: $$\frac{\left(x - \frac{ec}{e^2-1}\right)^2}{\left(\frac{c}{e^2-1}\right)^2} - \frac{y^2}{\left(\frac{c}{\sqrt{e^2-1}}\right)^2} = 1$$



$P = c/e$

perihelion $\dfrac{c}{1+e}$

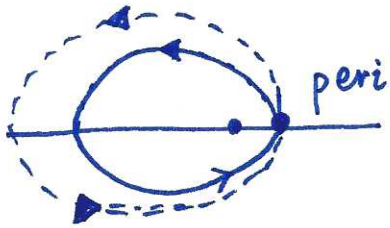
$a = \dfrac{c}{e^2-1}$

center $\left[\dfrac{ec}{e^2-1}, 0\right]$

define $\phi_0 = \cos^{-1} 1/e \Rightarrow$ r is finite when

$$\boxed{-(\pi - \cos^{-1}(1/e)) < \phi < \pi - \cos^{-1}(1/e)}$$

§ Change orbit



change from an elliptic orbit with (c_1, e_1) to another one with (c_2, e_2).

They are tagent at the prigee ⇒

$$\boxed{\frac{c_1}{1+e_1} = \frac{c_2}{1+e_2}}.$$

Define the thrust factor $\frac{v_2}{v_1} = \lambda$

$$\begin{cases} \lambda > 1 \Rightarrow \text{forward thrust} \\ 1 > \lambda > 0 \quad \text{backward thrust} \end{cases}$$

$l_2 = \lambda l_1$

Since $c = \frac{l^2}{\mu \gamma}$

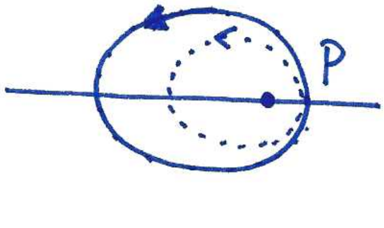
$\Rightarrow \quad c_2 = \lambda^2 c_1 \qquad \Rightarrow \quad \frac{1+e_2}{1+e_1} = \frac{c_2}{c_1} = \lambda^2$

or $\boxed{e_2 = \lambda^2 e_1 + (\lambda^2 - 1)}$

① If $\lambda > 1$, then $e_2 > e_1$. The two orbits have the same prigee the orbit becomes larger and more elliptical. At $e_2 \geq 1$, the orbit becomes open → parabola and hyperbala.

② If $\lambda < 1$, then $e_2 < e_1$. Then the new orbit becomes smaller and less elliptical. At $e_2 = 0$, the orbit becomes circluar.

how abont when $e_2 < 0$, then the equation of orbit changes to $r(\phi) = \frac{1}{1 - e_2 \cos\phi}$

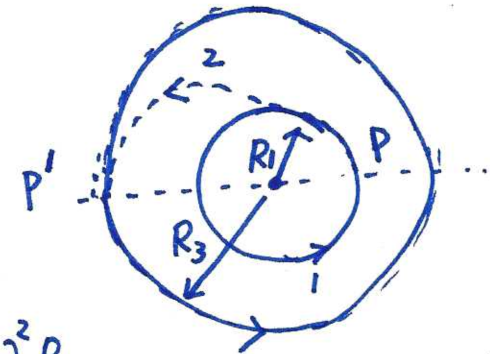


Then the prigee and apogee switch.

Changing between circular orbits.

The eccentricity of the orbit is

$e_1 = 0$, and $C_1 = R_1$



The eccentricity of the orbit 2 is e_2

$$\begin{cases} r = \dfrac{C_2}{1+e_2\cos\phi} \\ C_2 = \lambda^2 R_1 \end{cases} \Rightarrow \frac{C_2}{1+e_2} = \frac{\lambda^2 R_1}{1+e_2} = R_1 \Rightarrow e_2 = \lambda^2 - 1$$

and the apogee $\dfrac{C_2}{1-e_2} = R_3 \Rightarrow C_2 = R_3(1-e_2)$

$$\lambda^2 R_1 = R_3(2-\lambda^2)$$

$$\Rightarrow \boxed{\lambda^2 = \frac{2R_3}{R_1+R_3}} \quad \text{or } \lambda = \sqrt{\frac{2R_3}{R_1+R_3}}$$

The 2nd thrust. $\rightarrow \begin{cases} r = C_3 = R_3 \\ e_3 = 0 \end{cases} \quad C_3 = \lambda'^2 C_2$

$$\Rightarrow \lambda'^2 = \frac{C_3}{C_2} = \frac{R_3}{\lambda^2 R_1} = \frac{R_1+R_3}{2R_1} \quad \text{or} \quad \lambda' = \sqrt{\frac{R_1+R_3}{2R_1}}$$

The final speed and the initial speed

$$\begin{cases} v_3 = v_{2,app}\lambda' \\ \lambda v_1 = v_{2,peri} \end{cases} \quad \text{and } v_{2,app} \cdot R_3 = v_{2,peri} R_1$$

$$\Rightarrow v_3 = \lambda' \frac{v_{2,app}}{v_{2,per}} \cdot \lambda v_1 = \lambda'\lambda \frac{R_1}{R_3} v_1 = \sqrt{\frac{R_1}{R_3}}\, v_1$$

10
Cosmic velocities

Newton's solution to Kepler's problem paved the way for the space age, starting from thelaunch of Sputnik 1 in 1957. Below we explain the calculation of the three cosmic velocities. The first astronaut was Yuri Gagarin (1934-1968).

10.1 1st cosmic velocity - the orbiting velocity

The first cosmic velocity is that the an object does not fall on the group but orbiting around the earth.

$$m\frac{v_1^2}{R} = \frac{GMm}{R^2}$$
$$v_1^2 = \frac{GM}{R} \tag{10.1}$$

where m is the mass of the object, M is the earth mass, and R is the earth radius. Since $g = GM/R^2$, we arrive at

$$v_1 = \sqrt{Rg}, \tag{10.2}$$

and the period T is

$$T = 2\pi R/v = 2\pi\sqrt{R/g} \tag{10.3}$$

Plugging in $R = 6400km$ and $g \approx 10m/s^2$, we arrive at

$$v_1 \approx 8km/s, \quad T \approx 5024s \approx 84min. \tag{10.4}$$

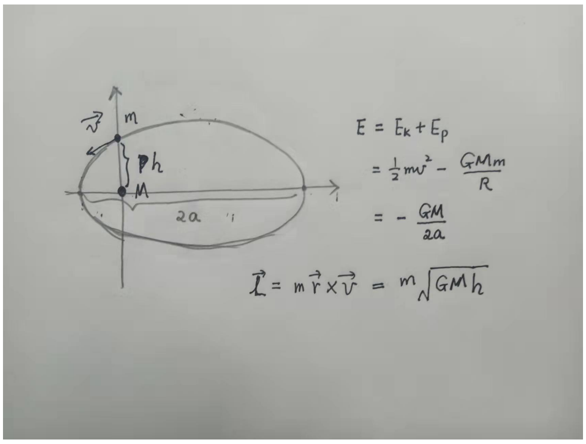


Figure 10.1 The total energy of an elliptic orbit is completely determined by the half major axis a as $E = -\frac{GMm}{2a}$. The angular momentum is completely determined by the half length of the cord passing the focus h as $l = m\sqrt{GMh} = mh\sqrt{GM/h}$.

10.2 A few useful results

The total energy is completely determined by a.

$$E = E_K + E_p = \frac{1}{2}mv^2 - \frac{GMm}{r} = -\frac{GMm}{2a}. \tag{10.5}$$

The angular momentum is completely determined by h.

$$\vec{L} = l\hat{z} = m\vec{r}\times\vec{v} = m\sqrt{GMh} = mh\sqrt{\frac{GM}{h}}. \tag{10.6}$$

For all the orbits with the same energy E, they share the same half major axis. But their orbital angular momentum is different. The circular orbit has the largest orbit angular momentum. This could be understood as follows: The kinetic energy only depends on speed but not the direction of the velocity, hence, if we let velocity be perpendicular to the radius, we can maximize angular momentum. This is just the circular orbit. For all the orbits with the same h, they share the same angular momentum, but their energies are different. The orbital angular momentum is $mrv\sin\theta$, hence, if $\theta = \pi/2$, we can let v be smallest, which leads to the minimum energy.

10.3 2nd cosmic velocity

The 2nd cosmic velocity v_2 refers to the minimal velocity at which the object can fly escaping from the earth. This means that the total energy, the sum of the kinetic and the gravity potential energy, is zero. Hence

$$\frac{1}{2}mv_2^2 - \frac{GMm}{R} = 0, \tag{10.7}$$

which shows that

$$v_2 = \sqrt{\frac{2GM}{R}}. \tag{10.8}$$

Its relation with the first cosmic velocity is

$$v_2 = \sqrt{2}v_1 \approx 11.2km/s. \tag{10.9}$$

At 2nd cosmic veolcity, the orbit is a parabola. Since the total energy is conserved at zero, this means that the satellite can move to infinity where $E_p = 0$ at which its velocity goes to zero. If $v \geqslant \sqrt{2GM/R}$, the orbit is a hyperbola. $E_{tot} = E_k + E_p > 0$, which means that the satellite can go to infinity with $E_k = \frac{1}{2}mv'^{,2} - GM/R$. Satellites with parabolic and hyperbolic orbits fly away and will not return to the earth.

10.4 3rd cosmic velocity

The 3rd cosmic velocity v_3 is considerably more complicated than the 1st and 2nd ones. This is the minimal velocity at which the object can escape the solar system.

First, we calculate the orbiting velocity of the earth. The earth-sun distance $R_e = 1.5 \times 10^8 km$, and the period is 1 year. Then the orbiting velocity of the earth around the Sun is

$$v_o = 2\pi R/T \approx 30km/s. \tag{10.10}$$

Then the escaping velocity respect to the Sun is

$$v_{es} = \sqrt{2}v_0 \approx 42.4km/s. \tag{10.11}$$

Nevertheless, the 3rd cosmic velocity is the object velocity when launched with respect to the earth surface, which can take the advantage of the earth orbiting velocity.

Let us consider three steps of launching a rocket to fly away from the earth. During these steps, the distance of the rocket with respect to the Sun changes very little, hence, its potential energy due to the gravity from the Sun can be approximately as a constant. We only count the kinetic energies of the rocket, the earth, the rocket-earth potential energy, and the chemical energy of the fuel.

The first stage is before the launch. The earth and the rocket have the same velocity v_0, and the energy stored in the chemical fuel E_c. The total energy is

$$E_1 = \frac{1}{2}(m + M)v_0^2 + E_{ch} - \frac{GMm}{R}. \tag{10.12}$$

The 2nd stage is that the rocket just acquires the 3rd cosmic velocity v_3 by burning out the chemical fuel, but is still very close to the earth surface. Then

$$E_2 = \frac{m}{2}(v_0 + v_3)^2 + \frac{M}{2}(v_0 + \Delta v)^2 - \frac{GMm}{R}, \tag{10.13}$$

where Δv is the recoil of the earth. According to momentum conservation, we have

$$\begin{aligned}(m + M)v_0 &= m(v_0 + v_3) + M(v_0 + \Delta v)\\ mv_3 + M\Delta v &= 0.\end{aligned} \tag{10.14}$$

Then Eq. 10.13 is reduced to

$$E_2 = \frac{1}{2}(m + M)v_0^2 + \frac{m}{2}v_3^2 + \frac{M}{2}\Delta v^2 - \frac{GMm}{R}. \tag{10.15}$$

The energy conservation $E_1 = E_2$ yields

$$E_{ch} = \frac{m}{2}(1 + \frac{m}{M})v_3^2 \approx \frac{m}{2}v_3^2, \tag{10.16}$$

which is correct to the zeroth order of m/M.

The 3rd stage is that the rocket flies away from the earth with the velocity $\sqrt{2}v_o$ with respect to the Sun. The total energy at this stagger is

$$E_3 = \frac{1}{2}m(\sqrt{2}v_o)^2 + \frac{M}{2}(v_o + \Delta v')^2, \tag{10.17}$$

where $\Delta v'$ is the recoil of the earth at the end of the 3rd stage. According to the momentum conservation,

$$\begin{aligned}(m + M)v_o &= m\sqrt{2}v_0 + M(v_o + \Delta v')\\ M\Delta v' &= -m(\sqrt{2} - 1)v_o\end{aligned} \tag{10.18}$$

Then Eq. 10.17 is reduced to

$$E_3 = \frac{m}{2}(\sqrt{2}v_o)^2 + \frac{M}{2}v_o^2 - m(\sqrt{2} - 1)v_o^2. \tag{10.19}$$

According to energy conservation, $E_1 = E_3$

$$\frac{m}{2}v_o^2\left((\sqrt{2})^2 - 2(\sqrt{2}-1) - 1\right) = \frac{m}{2}v_3^2 - \frac{GMm}{R}$$
$$v_3^2 = v_o^2\left(\sqrt{2}-1\right)^2 + v_2^2 \tag{10.20}$$

Plugging in $v_o = 30km/s$ and $v_2 = 11.2km/s$, then we arrive at

$$v_3 = \sqrt{30^2 \times 0.414^2 + 11.2^2} = 16.7km/s. \tag{10.21}$$