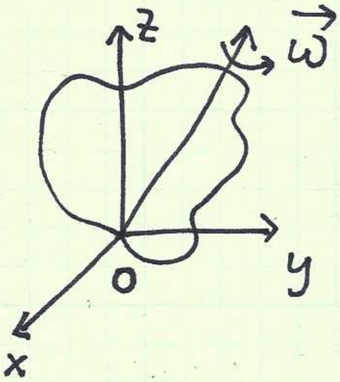


Fixed point rotation

§ definition: moment of inertial

Rotation around any axis

$$\vec{\omega} = (\omega_x, \omega_y, \omega_z)$$

$$\vec{L} = \sum_\alpha m_\alpha \vec{r}_\alpha \times \vec{v}_\alpha = \sum_\alpha m_\alpha \vec{r}_\alpha \times (\vec{\omega} \times \vec{r}_\alpha)$$

$$\vec{r} \times (\vec{\omega} \times \vec{r}) = \vec{\omega} r^2 - \vec{r}(\vec{\omega}\cdot\vec{r}) = (\omega_x r^2, \omega_y r^2, \omega_z r^2) - (\vec{\omega}\cdot\vec{r})(x, y, z)$$

$$= [(y^2+z^2)\omega_x - xy\,\omega_y - xz\,\omega_z,$$
$$-yx\,\omega_x + (z^2+x^2)\omega_y - yz\,\omega_z,$$
$$-zx\,\omega_x - zy\,\omega_y + (x^2+y^2)\omega_z]$$

$$\Rightarrow \begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

$$\Rightarrow \quad L_a = I_{ab}\,\omega_b$$

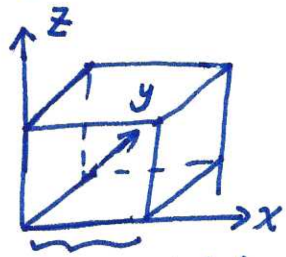
where $I_{ab} = \delta_{ab}\left(\sum_{\alpha=1}^{N} m_\alpha r_\alpha^2\right) - \sum_{\alpha=1}^{N} m_\alpha x_a x_b$

$$= \int dx\,dy\,dz\; \rho\,[r^2 \delta_{ab} - x_a x_b]$$

← $I_{ab} = I_{ba}$ symmetric tensor

Example. inertia tensor of a solid cube, rotating

① around the corner



a, total mass M

$$I_{xx} = \int_0^a dx \int_0^a dy \int_0^a dz \, \rho (y^2+z^2) = \rho a^2 \left[\frac{a^3}{3}\right] \times 2 = \frac{2a^2}{3} M$$

$$I_{yy} = I_{zz} = I_{xx}$$

$$I_{xy} = -\int_0^a dx \int_0^a dy \int_0^a dz \, \rho xy = -\rho \frac{a^2}{2} \frac{a^2}{2} \cdot a = -\frac{a^2}{4} M$$

$$\Rightarrow I = Ma^2 \begin{bmatrix} 2/3 & -1/4 & -1/4 \\ -1/4 & 2/3 & -1/4 \\ -1/4 & -1/4 & 2/3 \end{bmatrix}$$

If the cube is rotate around z-axis, $\vec{\omega} = \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix} \Rightarrow$

$$\vec{L} = Ma^2 \omega \begin{bmatrix} -1/4 \\ -1/4 \\ \frac{2}{3} \end{bmatrix}.$$

② if the cube rotates around it center

$$I_{xx} = \int_{-a/2}^{a/2} dx\, dy\, dz \, \rho (y^2+z^2) = \rho a^2 \frac{2}{3} \left(\frac{a}{2}\right)^3 \times 2 = \frac{a^2}{6} M$$

$$I_{yy} = I_{zz} = I_{xx}$$

$$I_{xy} = I_{yz} = I_{zx} = 0$$

$$\Rightarrow I = \frac{Ma^2}{6} \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \Rightarrow \vec{L} = \frac{Ma^2}{6} \vec{\omega}$$

"I" depends on the location of the origin & the orientation of the axis.

§ Kinetic energy of fix-point rotation

$$E_k = \frac{1}{2}\sum_\alpha m_\alpha \vec{v}_\alpha^{\,2}$$

$$\vec{v} = \vec{\omega}\times\vec{r} \Rightarrow v^2 = \omega^2 r^2 - (\vec{\omega}\cdot\vec{r})^2$$

$$= (\omega_x^2+\omega_y^2+\omega_z^2)(x^2+y^2+z^2) - (\omega_x x + \omega_y y + \omega_z z)^2$$

$$= \omega_x^2(y^2+z^2) + \omega_y^2(z^2+x^2) + \omega_z^2(x^2+y^2)$$

$$- 2\omega_x\omega_y xy - 2\omega_y\omega_z yz - 2\omega_z\omega_x zx$$

$$\Rightarrow E_k = \frac{1}{2}\sum_\alpha m_\alpha \vec{v}_\alpha^{\,2}$$

$$= \frac{1}{2}[\omega_x\ \omega_y\ \omega_z]\begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz}\end{bmatrix}\begin{bmatrix}\omega_x \\ \omega_y \\ \omega_z\end{bmatrix}$$

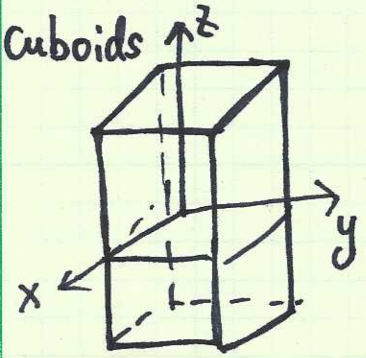
$$\Rightarrow E_k = \frac{1}{2}\omega_a I_{ab}\omega_b = \frac{1}{2}\vec{\omega}\cdot\vec{L}$$

I_{ab} is a 3×3 symmetric real matrix, which can be diagonalized by an orthogonal matrix. The diagonalized form of $I_{ab} \sim (\lambda_1, \lambda_2, \lambda_3)$ in the frame of principle axis.

$$\Rightarrow E_k = \frac{1}{2}(\lambda_1\omega_1^2 + \lambda_2\omega_2^2 + \lambda_3\omega_3^2)$$

$(\omega_1, \omega_2, \omega_3)$ is the projection of $\vec{\omega}$ along principle axis

Ex: Cuboids



$$I_{xy} = I_{yz} = I_{zx} = 0$$

$$I = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz}\end{bmatrix}$$

xyz are principle axis.

Euler equation

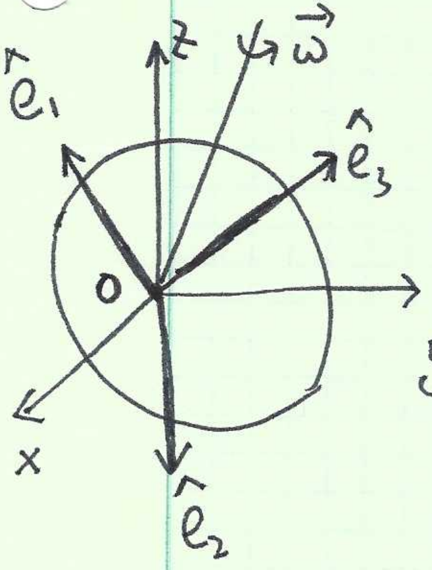
space frame : inertial frame with fixed xyz-axes

body frame : principle axes — body axes $\hat{e}_1, \hat{e}_2\, \hat{e}_3$

and the moment of inertial along the principle axes e1,e2,e3 are lambda1, lambda2, and lambda3, respectively.

rotating frame / non-inertial frame

If the body has a fixed point, we set the origin "O" at that point, then we do not need to worry the force exerted by "O". If the body has no fixed point, we choose the origin of the frames "O" at center of mass.



If the rigid body rotates at angular velocity $\vec{\omega}$.

Then we write down

$$\vec{\omega} = \omega_1 \hat{e}_1 + \omega_2 \hat{e}_2 + \omega_3 \hat{e}_3$$

in the body frame.

$$\vec{L} = \lambda_1 \omega_1 \hat{e}_1 + \lambda_2 \omega_2 \hat{e}_2 + \lambda_3 \omega_3 \hat{e}_3$$

$$\left(\frac{d\vec{L}}{dt}\right)_{space} = \vec{\Gamma} \leftarrow \text{torque} = \Gamma_1 \hat{e}_1 + \Gamma_2 \hat{e}_2 + \Gamma_3 \hat{e}_3$$

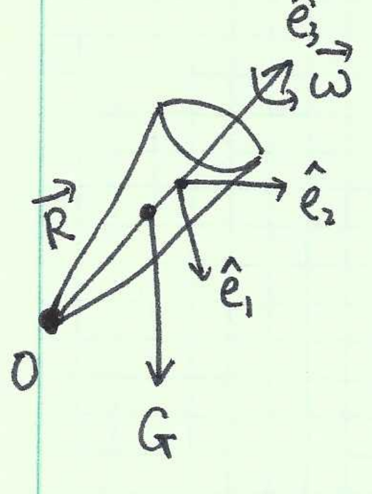
$$= \lambda_1 \dot{\omega}_1 \hat{e}_1 + \lambda_2 \dot{\omega}_2 \hat{e}_2 + \lambda_3 \dot{\omega}_3 \hat{e}_3 + \vec{\omega} \times (\lambda_1 \omega_1 \hat{e}_1 + \lambda_2 \omega_2 \hat{e}_2 + \lambda_3 \omega_3 \hat{e}_3) = \left(\frac{d\vec{L}}{dt}\right)_{body} + \vec{\omega} \times \vec{L}$$

$$\vec{\omega} \times \vec{L} = (\omega_1 l_2 - \omega_2 l_1)\hat{e}_3 + (\omega_2 l_3 - \omega_3 l_2)\hat{e}_1 + (\omega_3 l_1 - \omega_1 l_3)\hat{e}_2$$

$$= \omega_1\omega_2(\lambda_2-\lambda_1)\hat{e}_3 + \omega_2\omega_3(\lambda_3-\lambda_2)\hat{e}_1 + \omega_3\omega_1(l_1-l_3)\hat{e}_2$$

$$\Rightarrow \boxed{\begin{aligned} \lambda_1\dot{\omega}_1 - (\lambda_2-\lambda_3)\omega_2\omega_3 &= \Gamma_1 \\ \lambda_2\dot{\omega}_2 - (\lambda_3-\lambda_1)\omega_3\omega_1 &= \Gamma_2 \\ \lambda_3\dot{\omega}_3 - (\lambda_1-\lambda_2)\omega_1\omega_2 &= \Gamma_3 \end{aligned}}$$

Application: $\Gamma_{1,2,3}$ are components in the rotating frame, and thus is difficult to trace. Euler's equation is mostly useful for $\vec{\Gamma}=0$. or, for the case one of it's component is zero



symmetric top spinning in the gravity field

$$\vec{\Gamma} = \vec{R} \times \vec{G}$$

$$= R\,\hat{e}_3 \times \vec{G}$$

$\Rightarrow \vec{\Gamma}\cdot\hat{e}_3 = 0$, also because $\lambda_1 = \lambda_2$.

$$\Rightarrow \boxed{\lambda_3\dot{\omega}_3 = 0}$$

§ Euler equation with zero torque

$\vec{L}$ is conserved in the lab frame, but not in the body frame.

$$\boxed{\begin{aligned} \lambda_1 \dot{\omega}_1 &= (\lambda_2 - \lambda_3)\,\omega_2 \omega_3 \\ \lambda_2 \dot{\omega}_2 &= (\lambda_3 - \lambda_1)\,\omega_3 \omega_1 \\ \lambda_3 \dot{\omega}_3 &= (\lambda_1 - \lambda_2)\,\omega_1 \omega_2 \end{aligned}}$$

☆ if we start with $\vec{\omega}$ // a principle axis, say $\hat{e}_1$

$$\Rightarrow \omega_2 = \omega_3 = 0 \text{ at } t=0 \Rightarrow \begin{cases} \lambda_1 \dot{\omega}_1 = 0 \\ \lambda_2 \dot{\omega}_2 = 0 \\ \lambda_3 \dot{\omega}_3 = 0 \end{cases}$$

$$\Rightarrow \dot{\omega}_1 = \dot{\omega}_2 = \dot{\omega}_3 = 0$$

$\vec{\omega}$ is a constant.

☆ if $\vec{\omega}$ is not along any principle axis, and $\lambda_1 \neq \lambda_2 \neq \lambda_3$ then at most only one of $\omega_{1,2,3}$ can be nonzero, we have $\dot{\vec{\omega}} \neq 0$. Thus although $\vec{L}$ is conserved, but $\vec{\omega}$ is not!

☆ Next we study the stability ~~pro~~ of rotation around principle axis.

Suppose at $t=0$, $\vec{\omega} = \omega_3 \hat{e}_3$ with $\omega_1 = \omega_2 = 0$. Then we give a small perturbation ω_1 and ω_2, will $\omega_{1,2}$ grow larger and larger?

T: 0571-86886859
F: 0571-85271986
E: office@westlake.edu.cn
中国浙江省杭州市西湖区云栖小镇石龙山街18号，310024
18 Shilongshan Rd.,Cloud Town, Xihu District, Hangzhou,
Zhejiang Province, China, 310024

$$\lambda_3 \dot{\omega}_3 = (\lambda_1 - \lambda_2)\omega_1\omega_2 \sim \text{2nd order small quantity}$$
$$\approx 0,$$

i.e. $\omega_3 \sim$ const.

We then linearize the equation:
$$\begin{cases} \dot{\omega}_1 = \dfrac{\lambda_2-\lambda_3}{\lambda_1}\omega_3\,\omega_2 \\ \dot{\omega}_2 = \dfrac{\lambda_3-\lambda_1}{\lambda_2}\omega_3\,\omega_1 \end{cases}$$

try the solution:
$$\begin{cases} \omega_1(t) = A e^{\mu t} \\ \omega_2(t) = B e^{\mu t} \end{cases} \Rightarrow \begin{cases} A\mu = \dfrac{\lambda_2-\lambda_3}{\lambda_1}\omega_3 B & ① \\ B\mu = \dfrac{\lambda_3-\lambda_1}{\lambda_2}\omega_3 A & ② \end{cases}$$

if $A, B \neq 0 \Rightarrow \quad \mu^2 = \dfrac{\lambda_2-\lambda_3}{\lambda_1}\dfrac{\lambda_3-\lambda_1}{\lambda_2}\omega_3^2$

Hence ① if λ_3 lies between λ_1 and λ_2, then $\mu^2 > 0$

there exist a pair of positive and negative solutions
$$\mu_{1,2} = \pm\mu_0, \text{ with } \mu_0 = \omega_3\sqrt{\frac{(\lambda_2-\lambda_3)(\lambda_3-\lambda_1)}{\lambda_1\lambda_2}} > 0$$

then
$$\begin{cases} \omega_1(t) = A e^{\mu_0 t} + A' e^{-\mu_0 t} \\ \omega_2(t) = B e^{\mu_0 t} + B' e^{-\mu_0 t} \end{cases},$$

the coefficients A, B need to satisfy the relation $\dfrac{A}{B} = \dfrac{\lambda_2-\lambda_3}{\lambda_1}\dfrac{\omega_3}{\mu_0}$

and $A'/B' = -\dfrac{\lambda_2-\lambda_3}{\lambda_1}\dfrac{\omega_3}{\mu_0}$. Nevertheless, since $\mu_0 > 0$,

$\omega_1(t)$ and $\omega_2(t)$ diverge with time, and this rotation

is unstable!

西湖大學 WESTLAKE UNIVERSITY

T: 0571-86886859
F: 0571-85271986
E: office@westlake.edu.cn

中国浙江省杭州市西湖区云栖小镇石龙山街18号，310024
18 Shilongshan Rd.,Cloud Town, Xihu District, Hangzhou,
Zhejiang Province, China, 310024

② if λ_3 is the largest, or the smallest among λ_1, λ_2, and λ_3

then $\quad \mu = \pm i\mu_0 \quad , \quad \mu_0 = \frac{\omega_3}{\sqrt{\lambda_1\lambda_2}} \sqrt{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)}$

$$\begin{pmatrix} \omega_1(t) \\ \omega_2(t) \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix} e^{i\mu_0 t} \Rightarrow \quad A i\mu_0 = \frac{\lambda_2-\lambda_3}{\lambda_1}\omega_3 B$$

$$\Rightarrow B = \frac{i}{\sqrt{\lambda_1\lambda_2}} \frac{\lambda_1}{\lambda_2-\lambda_3} \sqrt{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)}$$

$$\Rightarrow \begin{pmatrix} \omega_1(t) \\ \omega_2(t) \end{pmatrix} = A \begin{pmatrix} 1 \\ \pm i\sqrt{\frac{\lambda_1(\lambda_1-\lambda_3)}{\lambda_2(\lambda_2-\lambda_3)}} \end{pmatrix} e^{i\mu_0 t} \qquad = \pm i\sqrt{\frac{\lambda_1}{\lambda_2}}\sqrt{\frac{\lambda_1-\lambda_3}{\lambda_2-\lambda_3}} \quad \left(\begin{matrix} + \text{ for } \lambda_2>\lambda_3 \\ - \text{ for } \lambda_2<\lambda_3 \end{matrix} \right.$$

Similarly another solution $\begin{pmatrix} \omega_1(t) \\ \omega_2(t) \end{pmatrix} = A' \begin{pmatrix} 1 \\ \mp i\sqrt{\frac{\lambda_1(\lambda_1-\lambda_3)}{\lambda_2(\lambda_2-\lambda_3)}} \end{pmatrix} e^{-i\mu_0 t}$

Hence, the general solution

$$\begin{pmatrix} \omega_1(t) \\ \omega_2(t) \end{pmatrix} = A \cdot \begin{pmatrix} 1 \\ \pm i\sqrt{\frac{\lambda_1}{\lambda_2}\frac{\lambda_1-\lambda_3}{\lambda_2-\lambda_3}} \end{pmatrix} e^{i\mu_0 t} + A' \begin{pmatrix} 1 \\ \mp i\sqrt{\frac{\lambda_1}{\lambda_2}\frac{\lambda_1-\lambda_3}{\lambda_2-\lambda_3}} \end{pmatrix} e^{-i\mu_0 t}$$

we want real solution $A' = A^* = \omega_0 e^{i\varphi}$

$$\Rightarrow \boxed{\begin{pmatrix} \omega_1(t) \\ \omega_2(t) \end{pmatrix} = \omega_0 \begin{pmatrix} \cos(\mu_0 t + \varphi) \\ \mp \sin(\mu_0 t+\varphi)\sqrt{\frac{\lambda_1}{\lambda_2}\frac{\lambda_1-\lambda_3}{\lambda_2-\lambda_3}} \end{pmatrix}}$$

which is a stable solution.

$-$ for $\lambda_2 > \lambda_3$

$+$ for $\lambda_2 < \lambda_3$

⊛ symmetric top, $\lambda_1 = \lambda_2$, then $\omega_3 = \text{const}$.

$$\begin{pmatrix} \omega_1(t) \\ \omega_2(t) \\ \omega_3 \end{pmatrix} = \begin{pmatrix} \omega_0 \cos\left(\frac{\omega_3(\lambda-\lambda_3)}{\lambda} t \right) \\ -\omega_0 \sin\left[\frac{\omega_3(\lambda-\lambda_3)}{\lambda} t \right] \\ \omega_3 \end{pmatrix}$$

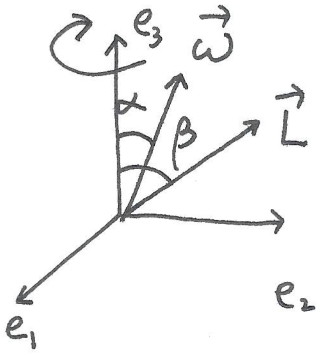
- eggshaped config $\lambda_3 < \lambda = \lambda_1 = \lambda_2$

$$\vec{\omega} = \omega_0 \cos\Omega_b t \, \hat{e}_1 - \omega_0 \sin\Omega_b t \, \hat{e}_2 + \omega_3 \hat{e}_3$$

$$\Omega_b = \frac{\omega_3}{\lambda}(\lambda - \lambda_3)$$

In the body frame $\vec{\omega}$ processes around $-\hat{e}_3$

$$\vec{L} = \lambda \omega_0 \cos\Omega_b t \, \hat{e}_1 - \lambda \omega_0 \sin\Omega_b t \, \hat{e}_2 + \lambda_3 \omega_3 \hat{e}_3$$



$\tan\alpha = \omega_0 / \omega_3$

$\tan\beta = \frac{\omega_0}{\omega_3} \frac{\lambda}{\lambda_3} \Rightarrow \beta > \alpha$

$\vec{\omega}$ and $\vec{L}$ precess around $\hat{e}_3$

$\vec{L}$ presesses around $-\hat{e}_3$ with the same angular velocity Ω_b.