General Physics I

Lect18. Blackbody Radiation

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Virial Theorem

- The Virial Theorem is useful when considering a collection of many particles and has special importance to central-force motion. For a general system of mass points with position vectors **r**$_i$ and applied forces **F**$_i$, consider the scalar product $G \equiv \sum_i \mathbf{p}_i \cdot \mathbf{r}_i$

$$\frac{dG}{dt} = \sum_i \mathbf{p}_i \cdot \dot{\mathbf{r}}_i + \sum_i \dot{\mathbf{p}}_i \cdot \mathbf{r}_i$$

Where $\sum_i \mathbf{p}_i \cdot \dot{\mathbf{r}}_i = \sum_i m\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i = \sum_i mv^2 = 2T$ and $\sum_i \dot{\mathbf{p}}_i \cdot \mathbf{r}_i = \sum_i \mathbf{F}_i \cdot \mathbf{r}_i$

Hence,

$$\frac{dG}{dt} = 2T + \sum_i \mathbf{F}_i \cdot \mathbf{r}_i$$

Time average of G

$$\frac{1}{\tau}\int_0^\tau \frac{dG}{dt}dt = \frac{G(\tau) - G(0)}{\tau} = \langle 2T \rangle + \left\langle \sum_i \mathbf{F}_i \cdot \mathbf{r}_i \right\rangle \to 0$$

$$\langle T \rangle = -\frac{1}{2}\left\langle \sum_i \mathbf{F}_i \cdot \mathbf{r}_i \right\rangle$$ (virial of the system)

Virial Theorem (continued)

- For a single particle subject to a conservative central force $F=-\nabla U$, the Virial theorem equals

$$\langle T\rangle = \frac{1}{2}\langle \nabla U\cdot \mathbf{r}\rangle = \frac{1}{2}\left\langle r\frac{\partial U}{\partial r}\right\rangle$$

If the potential is of the form $U=kr^{n+1}$ that is, $F=-k(n+1)r^{n}$, then $r\partial U/\partial r=(n+1)U$. Thus for a single particle in a central potential U, the Virial theorem reduces to

$$\langle T\rangle = \frac{n+1}{2}\langle U\rangle$$

*note T,U are kinetic energy and potential energy, respectively

Harmonic oscillator:
(linear force F=kr)

$$\langle T\rangle = +\langle U\rangle$$

In simple harmonic motion, the average kinetic and potential energies are both equal to half of the total energy.

Inverse-square law:
(gravity, Coulomb force etc., $F=k/r^2$)

$$\langle T\rangle = -0.5\langle U\rangle$$

In the Bohr model of the hydrogen atom, the kinetic energy of the bound electron has half of its potential energy. The same conclusion is true for the planetary motion in the solar system with gravitational force.

Charged oscillator—the case of an electron

- Suppose we have an electron oscillating up and down in an atom (charged oscillator), it radiates light. Its entire energy is **kT**---half kinetic, half potential.
- We want thermal equilibrium, so we enclose it in a box (of mirrors) so that the light comes back, "heating up" the atom.

- Next, determine how much light needed in such a box at temperature T. The EM radiation from the oscillation motion can be understood as a **damping term γ** using our knowledge from classical mechanics. The energy radiated per unit time is

$$\frac{dW}{dt} = \frac{\omega_0 W}{Q} = \frac{\omega_0 W \gamma}{\omega_0} = \gamma W.$$

- Now the oscillator should have an average energy kT, the average amount of energy radiated per unit time:

$$\langle dW/dt \rangle = \gamma kT$$

Light absorption cross section (derivation in Feynman Ch.32)

- Next we ask how much light must be shining on the oscillator.
- Let I(ω) be the intensity of light energy there is at frequency ω. How much radiation absorbed from a given incident light beam can be calculated it in terms of the product of I(ω) and the absorption cross section σ_s.

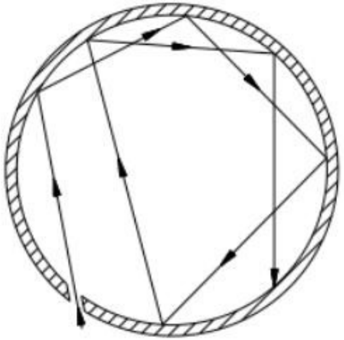
$$\sigma_s = \frac{8\pi r_0^2}{3}\left(\frac{\omega^4}{(\omega^2-\omega_0^2)^2+\gamma^2\omega^2}\right)$$

power

resonance

$$\sigma_s = \frac{2\pi r_0^2\omega_0^2}{3[(\omega-\omega_0)^2+\gamma^2/4]}$$ (at $\omega \to \omega_0$)

Where $r_0 = \frac{e^2}{m_e c^2} = 2.82 \times 10^{-15}$ m. is the classical electron radius.



I(ω) determines the color of a furnace at temperature T that we see when we open the door and look in the hole.

Radiation in equilibrium

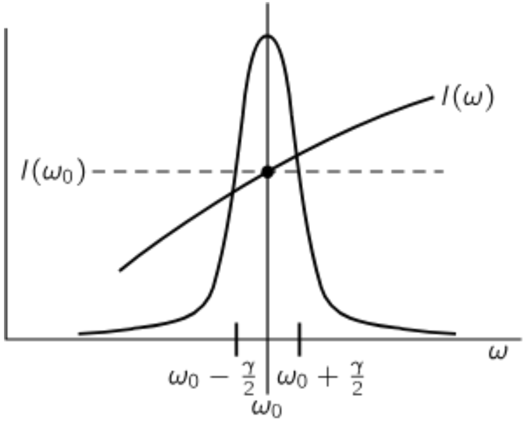
- Now we can equate the energy dissipated from oscillation and the energy returned as light absorption

$$\frac{dW_s}{dt} = \int_0^\infty I(\omega)\sigma_s(\omega)\, d\omega = \int_0^\infty \frac{2\pi r_0^2\omega_0^2 I(\omega)\, d\omega}{3[(\omega-\omega_0)^2+\gamma^2/4]}$$

- Since the electron can be viewed as an oscillator that can freely move in *3D*, so that it does not restricted to a particular polarization of light, the energy loss per unit time is $3\gamma kT$, hence

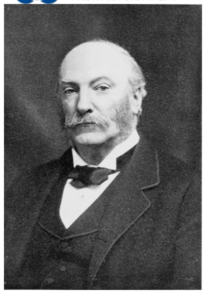
$$\tfrac{2}{3}\pi r_0^2\omega_0^2 I(\omega_0)\int_0^\infty \frac{d\omega}{(\omega-\omega_0)^2+\gamma^2/4} = 3\gamma kT$$

solution $I(\omega_0) = \frac{9\gamma^2 kT}{4\pi^2 r_0^2\omega_0^2} \xrightarrow[\gamma = \frac{\omega_0}{Q} = \frac{2}{3}\frac{r_0\omega_0^2}{c}]{r_0 = \frac{e^2}{m_e c^2}} \boxed{I(\omega) = \frac{\omega^2 kT}{\pi^2 c^2}}$



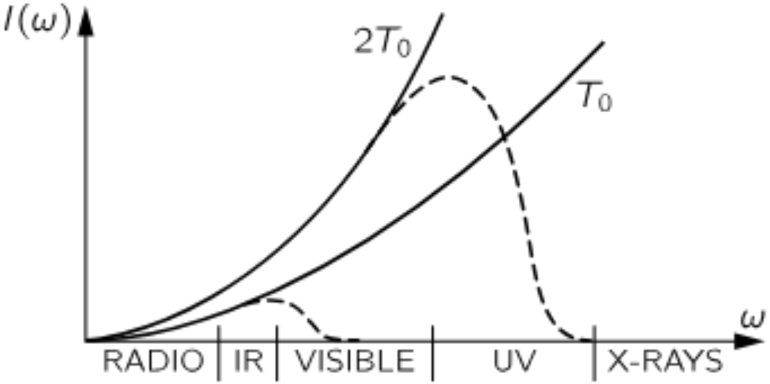
Q for a radiating oscillator is about 10^8, therefore we can we simply take the I(ω) outside the integral sign and replace it with $I(\omega_0)$.

A cloud of 20th century physics: blackbody radiation



Lord Rayleigh

$$I(\omega) = \frac{\omega^2 kT}{\pi^2 c^2}$$



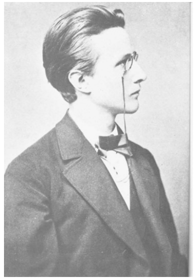
The blackbody intensity according to classical physics (solid curves) vs. the actual distribution (dashed curves).

- This equation gives us the distribution of light in a hot furnace, the ***blackbody radiation***.
- The good part, it is independent of the properties of the oscillator (charge, mass, etc.) , which only depend on the universal property at *equilibrium*—the ***temperature***.
- The bad part, it predicts a lot of x-rays in such black box, and that the total energy, integrated over all frequencies, is infinity!

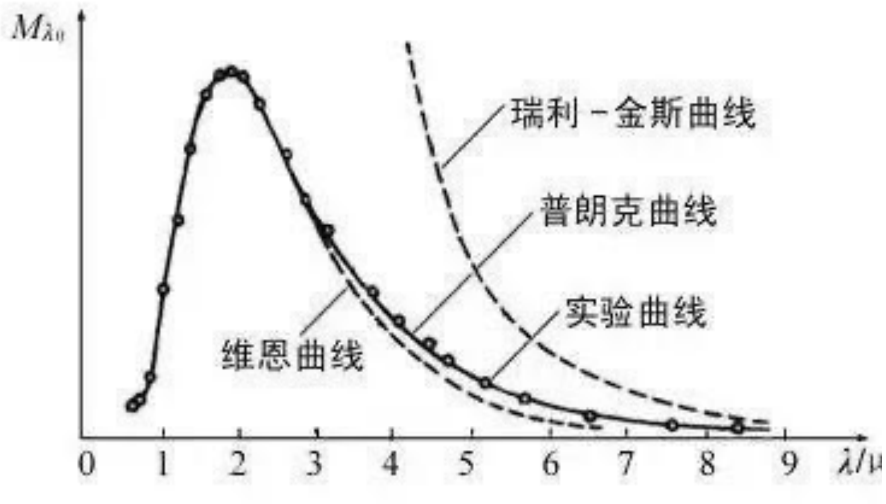
Incapability of classical theory

Equipartition and the quantum oscillator

Max Planck found the *right* empirical formula to replace the kT, such that the function fits the curves from experiments. However, this takes a bold assumption, that the harmonic oscillator can take up energies only $\hbar\omega$ at a time, not any value.



Max Planck



Quantum mechanics replaces the probability $e^{-\mathrm{Energy}/kT}$ of continuous energy classically with discrete steps of energy excess ΔE, so the probability goes down as $e^{-\Delta E/kT}$.

In 1900, Planck presented this idea at *Deutsche Physikalische Gesellschaft*, marking the beginning of the end of classical mechanics.

Planck's law

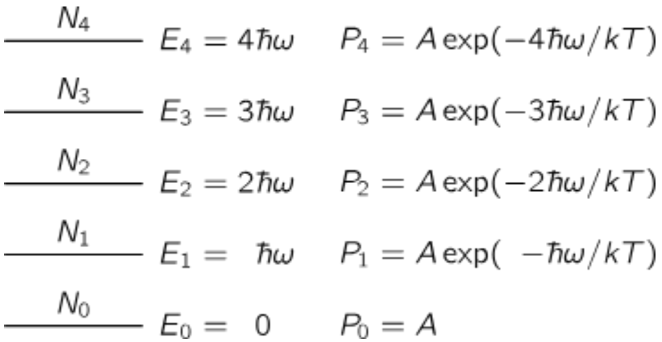
Let us make $x=e^{-\hbar\omega/kT}$, $N_1=N_0x$, $N_2=N_0x^2 \ldots N_n=N_0x^n$, and energy of each state of the oscillator is $E_n=n\hbar\omega$. The average energy of the oscillator is

$$\langle E\rangle = \frac{E_{\text{tot}}}{N_{\text{tot}}} = \frac{N_0\hbar\omega(0+x+2x^2+3x^3+\cdots)}{N_0(1+x+x^2+x^3+\cdots)}$$

$$= \frac{\hbar\omega\sum nx^n}{\sum x^n} = \frac{\hbar\omega\, x d(\sum x^n)/dx}{\sum x^n} = \hbar\omega\frac{xd[ln(\sum x^n)]}{dx}$$

$$= \hbar\omega\frac{xd[-\ln(1-x)]}{dx} = \hbar\omega x/(1-x)$$

In a quantum oscillator $\langle E\rangle = \dfrac{\hbar\omega}{e^{\hbar\omega/kT}-1}.$



$E_4 = 4\hbar\omega$ $\quad P_4 = A\exp(-4\hbar\omega/kT)$

$E_3 = 3\hbar\omega$ $\quad P_3 = A\exp(-3\hbar\omega/kT)$

$E_2 = 2\hbar\omega$ $\quad P_2 = A\exp(-2\hbar\omega/kT)$

$E_1 = \hbar\omega$ $\quad P_1 = A\exp(-\hbar\omega/kT)$

$E_0 = 0$ $\quad P_0 = A$

Now replace the classical "kT" with our new "quantum energy"

$$I(\omega) = \frac{\omega^2 \cancel{kT}}{\pi^2c^2}$$

We have derived the very first quantum-mechanical formula.

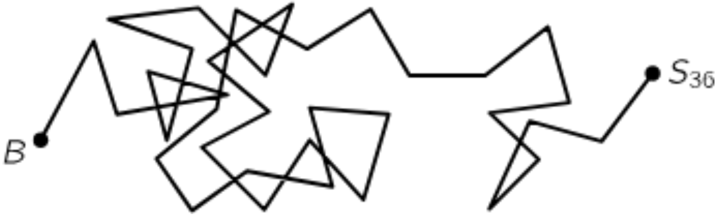
$$I(\omega)\,d\omega = \frac{\hbar\omega^3\,d\omega}{\pi^2c^2\,(e^{\hbar\omega/kT}-1)}$$

Random walk

In 1827, botanist Robert Brown discovered the Brownian movement, which he attributed to tiny plant pollen particles jiggling in water. The collisions are random, making it impossible to know where the particle is after a long time. But we can still ask, on the average, how far away will it travel?

Assume at N step, it moves from **R**$_{N-1}$ to **R**$_N$, with a vector **L**, s.t. $\boldsymbol{R}_N = \boldsymbol{R}_{N-1} + \boldsymbol{L}$

$$\boldsymbol{R}_N \cdot \boldsymbol{R}_N = R_N^2 = R_{N-1}^2 + 2\boldsymbol{R}_{N-1} \cdot \boldsymbol{L} + L^2$$



Therefore, the average

$$\langle R_N^2 \rangle = \langle R_{N-1}^2 \rangle + L^2$$

And by induction,

$$\langle R_N^2 \rangle = NL^2$$

We can understand this as the random vector, L, adds up quadratically, not linearly (property of noise)

Time dependence of a random walk

To add the time aspect, we need to consider the equation of motion, where μ can be determined directly from experiment

$$m\frac{d^2x}{dt^2}+\mu\frac{dx}{dt}=F_{\text{ext}}\xrightarrow{\text{Time average}}\left\langle mx\frac{d^2x}{dt^2}\right\rangle+\mu\left\langle x\frac{dx}{dt}\right\rangle=\langle xF_x\rangle$$

Use $mx\frac{d^2x}{dt^2}=m\frac{d[x(dx/dt)]}{dt}-m\left(\frac{dx}{dt}\right)^2$

We have $-\langle mv^2\rangle+\frac{\mu}{2}\frac{d}{dt}\langle x^2\rangle=0 \longrightarrow \frac{d\langle x^2\rangle}{dt}=2\frac{kT}{\mu}$

The average squared distance traveled with random walk in 3 d.o.f., after time t

$$\langle R^2\rangle=6kT\frac{t}{\mu}$$

We used this to determine the Boltzmann's constant in old days