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Lect 2: Vectors, scalar and cross products, rotations

Outline:

① Definition of vector, parallelogram law

② inner product

③ Cross product – directed area

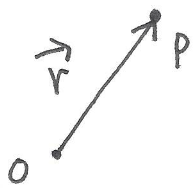
④ Frame and coordinate transformations

⑤ invariance of the scalar product.

§ vector:

Various physical quantities have both direction and magnitude, which are represented as vectors. For example, the displacement, velocity, acceleration. force, etc, are all vectors. Pictorially, vectors can be represented by a line segment with a direction.

①

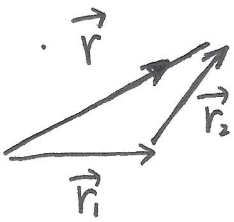


the direction of $\vec{r}$ is often denoted as $\hat{r}$. The length of $\hat{r}$ is 1, hence, $\hat{r}$ is often called the unit vector.

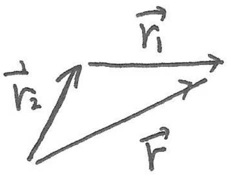
$-\vec{r}$ has the opposite direction but the same magnitude as $\vec{r}$

we have $\vec{r} + (-\vec{r}) = \vec{r} - \vec{r} = 0.$

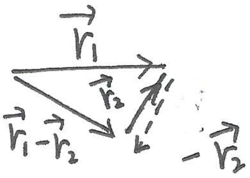
② Summation of two vectors



$\vec{r} = \vec{r}_1 + \vec{r}_2$ parallelogram law of addition



$\vec{r} = \vec{r}_2 + \vec{r}_1 = \vec{r}_1 + \vec{r}_2$



$\vec{r}_1 + (-\vec{r}_2) = \vec{r}_1 - \vec{r}_2$

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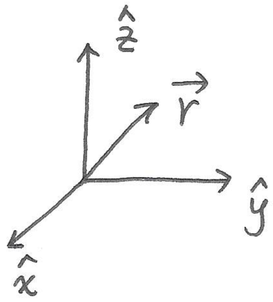
③ Components of a vector



Let $\hat{x}, \hat{y}, \hat{z}$ be a set of orthogonal unit vectors.

They define a cartesian coordinate system.

A vector $\vec{r}$ is written as

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z},$$ where $x, y,$ and z are called the components. The magnitude of $\vec{r}$ is denoted as r or $|r|$.

$$r = \sqrt{x^2 + y^2 + z^2}.$$

④ inner product. (scal~~ar~~ product)

Consider two vectors $\vec{r}_1 = x_1\hat{x} + y_1\hat{y} + z_1\hat{z}$, and $\vec{r}_2 = x_2\hat{x} + y_2\hat{y} + z_2\hat{z}$.

Define their inner product as

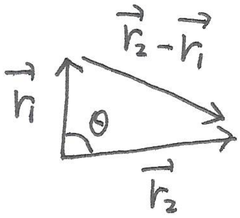
$$\boxed{\vec{r}_1 \cdot \vec{r}_2 = x_1 x_2 + y_1 y_2 + z_1 z_2.}$$

$$\hat{x}\cdot\hat{x} = \hat{y}\cdot\hat{y} = \hat{z}\cdot\hat{z} = 1$$

$$\begin{cases} \hat{x}\cdot\hat{y} = \hat{y}\cdot\hat{x} = 0 \\ \hat{x}\cdot\hat{z} = \hat{z}\cdot\hat{x} = 0 \\ \hat{y}\cdot\hat{z} = \hat{z}\cdot\hat{y} = 0 \end{cases}$$

then $r^2 = \vec{r}\cdot\vec{r}$

⑤ Geometrical meaning of the inner product

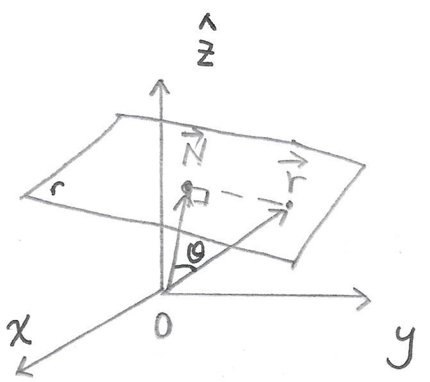


$$|\vec{r}_2 - \vec{r}_1|^2 = (\vec{r}_2 - \vec{r}_1)\cdot(\vec{r}_2 - \vec{r}_1)$$

$$= \vec{r}_2\cdot\vec{r}_2 + \vec{r}_1\cdot\vec{r}_1 - 2\vec{r}_1\cdot\vec{r}_2 = r_2^2 + r_1^2 - 2r_1 r_2\cos\theta$$

$$\Rightarrow \boxed{\vec{r}_1\cdot\vec{r}_2 = r_1 r_2 \cos\theta}$$

Applications of the inner product



① Equation of a plane

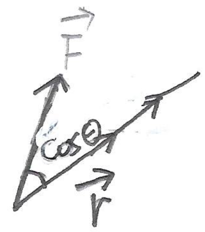
$\vec{ON}$ is the normal to the plane with the foot N located in the plane. $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$ is an arbitary point on the plane.

$$\vec{r} \cdot \vec{ON} = r \cdot |ON| \cos\theta = |ON|^2$$, denote $\vec{ON} = N_x \hat{x} + N_y \hat{y} + N_z \hat{z}$

$$x \cdot N_x + y N_y + z N_z = |ON|^2$$

i.e. $$\boxed{\frac{x\ N_x}{|ON|^2} + \frac{y \cdot N_y}{|ON|^2} + \frac{z N_z}{|ON|^2} = 1.}$$

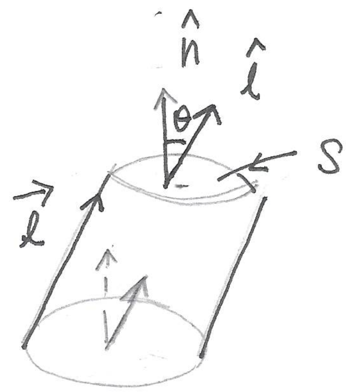
② Work



$$W = Fr\cos\theta = \vec{F} \cdot \vec{r}$$

$$P = dw/dt = \vec{F} \cdot d\vec{r}/dt = \vec{F} \cdot \vec{v}$$

③ Volume swept by an area



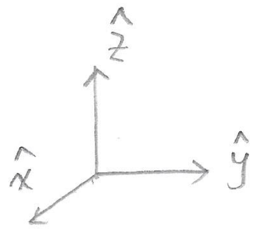
$$\vec{S} = S\hat{n} \quad \text{(directed area)}$$

The volume $= \vec{S} \cdot \vec{l} = S\,\hat{n} \cdot \vec{l}$

§ Cross product:

We define the cross product for the basis vectors as

$$\boxed{\begin{array}{lll} \hat{x}\times\hat{x} = \hat{y}\times\hat{y} = \hat{z}\times\hat{z} = 0 & & \\ \hat{x}\times\hat{y} = \hat{z}, & \hat{y}\times\hat{z} = \hat{x}, & \hat{z}\times\hat{x} = \hat{y}, \\ \hat{y}\times\hat{x} = -\hat{z}, & \hat{z}\times\hat{y} = -\hat{x}, & \hat{x}\times\hat{z} = -\hat{y}. \end{array}}$$



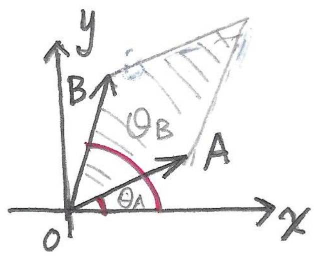
Right-hand-thread rule

Consider two vectors $\vec{A} = A_x\hat{x} + A_y\hat{y}$. $\vec{B} = B_x\hat{x} + B_y\hat{y}$, then

$$\vec{A}\times\vec{B} = (A_x\hat{x} + A_y\hat{y})\times(B_x\hat{x} + B_y\hat{y}) = (A_xB_y - A_yB_x)\,\hat{z}$$

$A_x = |OA|\cos\theta_A \quad A_y = |OA|\sin\theta_A$

$B_x = |OB|\cos\theta_B \quad B_y = |OB|\sin\theta_B$



$$\vec{A}\times\vec{B} = |OA|\cdot|OB|\,(\cos\theta_A\sin\theta_B - \sin\theta_A\cos\theta_B)\,\hat{z}$$

$$= |OA|\cdot|OB|\,\sin(\theta_B - \theta_A)\,\hat{z}$$

Hence, the directed area of a parallegram is $\vec{A}\times\vec{B}$, whose direction is perpendicular to the plane following the right-hand-thread rule.

This conclusion is also true for the general case of vectors $\vec{A}$ and $\vec{B}$.

- Assume $\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$, $\vec{B} = B_x\hat{x} + B_y\hat{y} + B_z\hat{z}$

Then
$$\boxed{\vec{C} = \vec{A}\times\vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \hat{x}(A_yB_z - A_zB_y) + \hat{y}(A_zB_x - A_xB_z) + \hat{z}(A_xB_y - A_yB_x)}$$

please notice the cyclic (rotation) pattern of indices.

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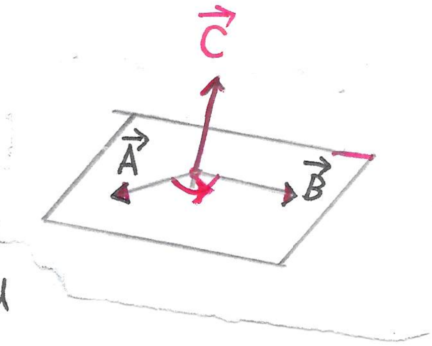
$$\vec{C}\cdot\vec{A} = A_x(A_yB_z - A_zB_y) + A_y(A_zB_x - A_xB_z) + A_z(A_xB_y - A_yB_x) = 0$$

or $\vec{C}\cdot\vec{A} = \begin{vmatrix} A_x & A_y & A_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = 0$

Similarly $\vec{C}\cdot\vec{B} = 0$ $\Rightarrow$ $\vec{C} \perp$ plane spanned by $\vec{A}, \vec{B}$



The direction of $\vec{C}$ follows the right-hand rule for the specical case discussed for $\vec{A}.\vec{B}$ lying in the xy plane. Since the right or left-hand convention cannot be changed smoothly, the right-hand convention is mantained for the general case.

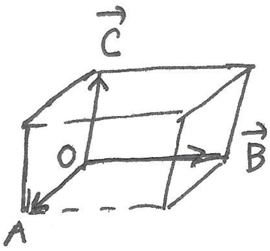
$\boxed{\cdot\ |C|^2 = ?}$

$$|C|^2 = (A_yB_z - A_zB_y)^2 + (A_zB_x - A_xB_z)^2 + (A_xB_y - A_yB_x)^2$$
$$= (A_x^2 + A_y^2 + A_z^2)(B_x^2 + B_y^2 + B_z^2) - (A_xB_x + A_yB_y + A_zB_z)^2$$
$$= |A|^2|B|^2 - |A\cdot B|^2 = |A|^2|B|^2(1-\cos^2\theta) = |A|^2|B|^2\sin^2\theta$$

Hence $\vec{C}$ is the direted area of the parallagram $\vec{A}\times\vec{B}$ for the general case!

- It's easy to check that $\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$. The direction of the area ~~is~~ $\vec{B}\times\vec{A}$ is opposite to that of $\vec{A}\times\vec{B}$.

- Volume of a parrallelpipeed



$$\vec{S}_{AB} \cdot \vec{OC} = (\vec{OA} \times \vec{OB}) \cdot (\vec{OC})$$

or simply $(\vec{A} \times \vec{B}) \cdot \vec{C}$.

We could also interprete this volume as $\vec{S}_{BC} \cdot \vec{OA} = (\vec{B} \times \vec{C}) \cdot \vec{A}$

and $\vec{S}_{\vec{C}\vec{A}} \cdot \vec{OB} = (\vec{C} \times \vec{A}) \cdot \vec{B}$.

The scalar triple is invariant under cyclically permutation.

$$(\vec{A} \times \vec{B}) \cdot \vec{C} = (\vec{B} \times \vec{C}) \cdot \vec{A} = (\vec{C} \times \vec{A}) \cdot \vec{B}$$

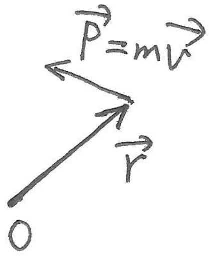
$$= -(\vec{B} \times \vec{A}) \vec{C} = -(\vec{C} \times \vec{B}) \cdot \vec{A} = -(\vec{A} \times \vec{C}) \cdot \vec{B}$$

"$-$" sign appears for exchanging two vectors.

- examples:

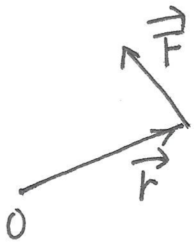
① angular momentum

$$\vec{L} = \vec{r} \times \vec{p} = m \vec{r} \times \vec{v}$$



② torque

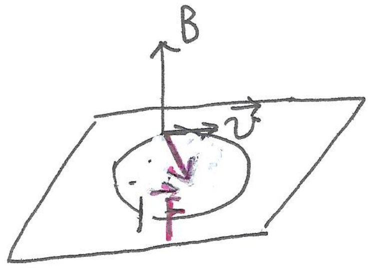
$$\vec{N} = \vec{r} \times \vec{F}$$



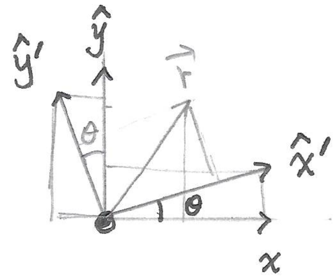
③ Lorentz force

$$\vec{F} = \frac{q}{c} \vec{v} \times \vec{B} \quad \text{(Gaussian)}$$

$$= q \vec{v} \times \vec{B} \quad \text{(SI)}$$



§ Frame transformation — passive view

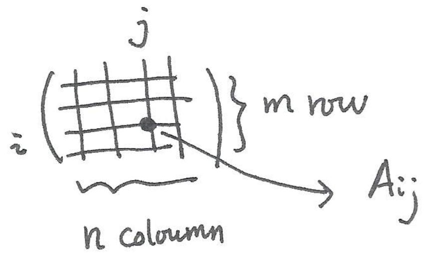


Consider two frames with the bases vectors $(\hat{x}', \hat{y}', \hat{z}')$ and $(\hat{x}, \hat{y}, \hat{z})$ with $\hat{z} = \hat{z}'$, but $\hat{x}, \hat{y}$ and $\hat{x}', \hat{y}'$ are rotated at the angle of θ.

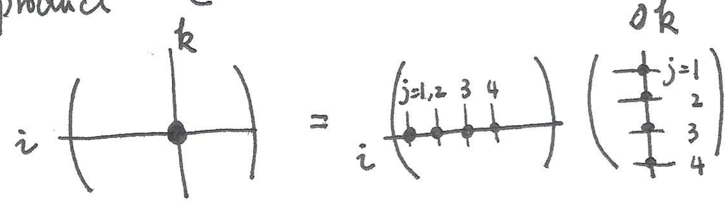
$$\hat{x}' = \hat{x}\cos\theta + \hat{y}\sin\theta$$
$$\hat{y}' = -\hat{x}\sin\theta + \hat{y}\cos\theta$$

$$\Rightarrow \boxed{(\hat{x}', \hat{y}') = (\hat{x}, \hat{y}) \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}$$

Matrix

Matrix $A =$ 

matrix product $C = AB$



$$\boxed{C_{ik} = \sum_j A_{ij} B_{jk}}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg, & af+bh \\ ce+dg, & cf+dh \end{pmatrix}$$

$$\neq$$

$$\begin{pmatrix} e & f \\ g & h \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ae+fc, & eb+df \\ ag+hc, & bg+dh \end{pmatrix}$$

$$(A^T)_{ij} = A_{ji}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

$\boxed{T \rightarrow \text{transpose}}$

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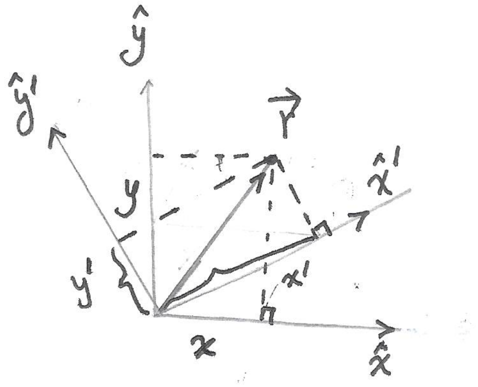
★ Coordinate transframation:

$$\vec{r} = x\hat{x} + y\hat{y} = (\hat{x}, \hat{y})\begin{pmatrix} x \\ y \end{pmatrix}$$

$$= x'\hat{x}' + y'\hat{y}' = (\hat{x}', \hat{y}')\begin{pmatrix} x' \\ y' \end{pmatrix} = (\hat{x}, \hat{y})\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix}$$



$$\Rightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$(x', y') = (x, y)\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

★ invariance of the inner product

$\vec{r}_1 = \hat{x}x_1 + \hat{y}y_1$, $\vec{r}_2 = \hat{x}x_2 + \hat{y}y_2 \Rightarrow$ under the basis $(\hat{x}, \hat{y})$

we have $\vec{r}_1 \cdot \vec{r}_2 = x_1x_2 + y_1y_2 = (x_1\ y_1)\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$

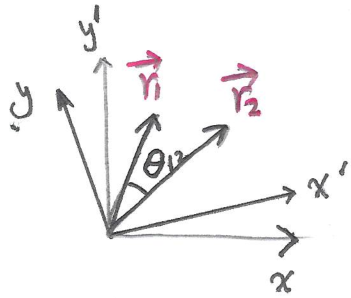
Similarly, under the basis of $(\hat{x}', \hat{y}')$, we write

$$\vec{r}_1 = \hat{x}'x_1' + \hat{y}'y_1',\quad \vec{r}_2 = \hat{x}'x_2' + \hat{y}'y_2' \Rightarrow \vec{r}_1\cdot\vec{r}_2 = (x_1', y_1')\begin{pmatrix} x_2' \\ y_2' \end{pmatrix}$$

Are they consistent? Yes, otherwise it does not look good.

Proof: $$(x_1'\ y_1')\begin{pmatrix} x_2' \\ y_2' \end{pmatrix} = (x_1 y_1)\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

$$= (x_1\ y_1)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = (x_1\ y_1)\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$



This makes sense that $\vec{r}_1\cdot\vec{r}_2 = |r_1||r_2|\cos\theta_{12}$, which should be independent of frame transformation, and this is why

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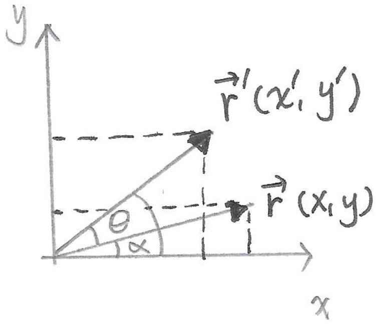
$\vec{r}_1 \cdot \vec{r}_2$ is called the scalar product:

$\boxed{\text{scalar: a quantity is invariant under the frame transformation.}}$

- Initiative viewpoint: fix frame, but rotate the vector

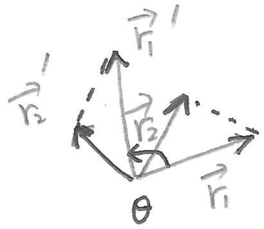
$$x' = r\cos(\theta+\alpha) = r(\cos\theta\cos\alpha - \sin\theta\sin\alpha)$$
$$= \cos\theta\, x - \sin\theta\, y$$
$$y' = r\sin(\theta+\alpha) = r(\sin\theta\cos\alpha + \cos\theta\sin\alpha)$$
$$= \sin\theta\, x + \cos\theta\, y$$



$$\Rightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

consider a pair of vectors $\vec{r}_1, \vec{r}_2$

rotation around the $\hat{z}$-axis at the angle of θ



we arrive at $\vec{r}_1', \vec{r}_2'$,

then it's easy to show $\vec{r}_1' \cdot \vec{r}_2' = \vec{r}_1 \cdot \vec{r}_2$.

We denote transformation matrix $U = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

$$\vec{r}_1' \cdot \vec{r}_2' = (x_1', y_1') \begin{pmatrix} x_2' \\ y_2' \end{pmatrix} = (x_1, y_1) \boxed{U^T U} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

Please check $\boxed{U^T U = I}$

orthogonal matrix.