西湖大學 WESTLAKE UNIVERSITY

Lecture 4: Newton's laws of motion (I)

Outline:

⊛ Newton's standing on Galileo's shoulders

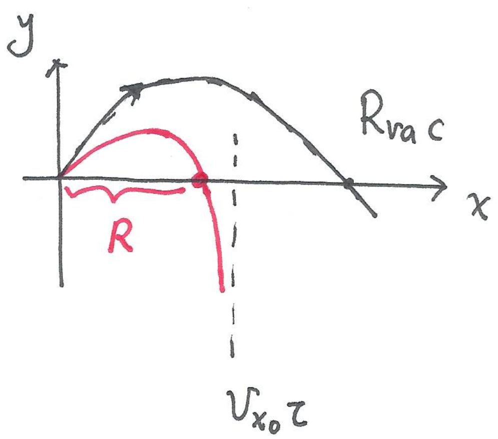
law of inertial

Newton's 2nd law — concepts of force, mass

⊛ Projectile motion with resistance

linear v.s. quadratic drags

$$\begin{cases} m\dot{v}_x = -b v_x \\ m\dot{v}_y = -mg - b v_y \end{cases} \qquad \tau = m/b \qquad v_\infty = mg/b$$



$$R \approx R_{vac}\left[1 - \frac{4}{3}\frac{v_{y_0}}{v(\infty)}\right]$$

* Newten was remarkable! — more progress then Galileo.

Galileo: discovery of the principle of inertia (Newton's 1st law of motion). If a body is not exterted a force on it, it keeps still, or it maintains the motion at a constant velocity.

Hower, this law is more like a definition. It's valid in special frames called inertial frames, but not in all the frames. For example, in the frame of an accelerated train, a free body will also accelerate. But then how to judge a frame is inertial? We will say, ok, look at a body, if it has no force acting on it, and it's either at rest or moves with a constant speed, then the frame is inertial! This sounds like a circular reasoning! Hence, the true meaning of "the law of inertia" is the assumption of the existence of the inertial frame! It's role is like the 5 prepositions of Euclid "elements".

Galileo also realized that force results in acceleration, and acceleration is proportional to force. But he did not propose the concept of mass, and the concept of force was concrete, such as gravity of the free fall, and other daily life examples.

Standing on the shoulders of Galileo, Newton improved the concept of force to a much more general and abstract level. Using his imagination, Newton generalized the gravity for falling apples to the motions of moon, planets, etc! Newton also proposed the concept of "mass", as a measurment of inertia. Then we can compare the accelerations among different objects and different forces!

> Newton's 2nd law: The time-rate-of-change of the momentum is proportional to force! $\vec{F} = \frac{d}{dt}(m\vec{v})$.

For non-relativistic physics, $v \ll c$, the mass can be approximated as a constant. In this case, Newton's 2nd law becomes

$$\vec{F} = m\frac{d\vec{v}}{dt} = m\vec{a} = m\frac{d^2\vec{r}}{dt^2}$$

This is a 2nd order differential equation, and also a vector equation. Acceleration not only could mean the change of speed, but also could mean the change of motion direction!

(*) Example — projectile motion with resistance

If we have the knowledge of force, in principle, we should be able to solve its equation of motion. Now we consider a more realistic projectile motion by considering the air resistance $\vec{f} = -f(v)\hat{v}$, which is always in the opposite direction of velocity causing dissipation. (The dissipation power $\vec{f}\cdot\vec{v} < 0$).

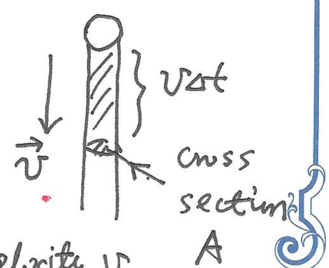
When the speed v is small, f is proportional to v, which is called the stoke's law of viscosity.

$$\boxed{f_{lin} = 3\pi\eta D v,}$$

where η is the viscocity of the fluid, D is the droplet diameter

(The unit of η is $kg/(m\cdot s)$].

Actually the resistance also contains the quadrtic part



In a short time Δt, the projectile travels at a distance $v\Delta t$. It pushes the air of the volume $vA\Delta t$ to the velocity v.

κ is a coefficient, $0<\kappa<1$.

$$f_q \cdot \Delta t = \kappa \rho_{air} v A \Delta t \cdot v \Rightarrow \boxed{f_q = \kappa \frac{\pi}{4} D^2 \rho_a v^2}$$

which part is more significant?

$$f_q / f_{lin} = \frac{\kappa\pi \quad D^2 \rho_a v^2}{4 \cdot 3\pi \, \eta D v} = \frac{\kappa}{12} \boxed{\frac{D v \rho_a}{\eta}}$$

Reynolds number

plug in $\eta = 1.7 \times 10^{-5}$ kg/(m·s), $\rho_a = 1.29$ kg/m³

$$\Rightarrow f_q/f_{lin} = \left[6\times10^{3} \frac{s}{m^2}\right] \kappa \, D v .$$

We chose a commonly used $\kappa = 1/4$

$$\rightarrow 1.5\times10^{3} (s/m^2) \, D v$$

- base ball $D = 7$ cm and $v = 5$ m/s $\Rightarrow f_q/f_{lin} = 600$
- rain droplet $D = 1$ mm. $v = 0.6$ m/s $\quad f_q/f_{lin} \simeq 1$
- milikan oil drop $D = 1.5\,\mu$m. $v = 5\times10^{-5}$ m/s $\quad f_q/f_{lin} \simeq 10^{-7}$.

For big and fast projectiles, the qudratic drag is more important, while for small and slow projectiles, the linear one is more important.

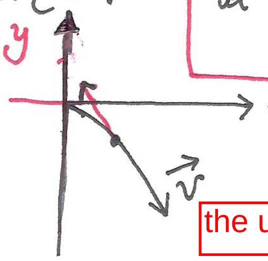
A: Motion with the linear air resistance

Newton's notation

$$\boxed{\begin{aligned} \frac{d}{dt} A &\Longrightarrow \dot{A} \\ \frac{d^2}{dt^2} A &\rightarrow \ddot{A} \end{aligned}}$$

$$m\ddot{\vec{r}} = m\vec{g} - b\vec{v}, \quad \text{where } b = 3\pi\eta D.$$
$$= m\dot{\vec{v}}$$

$$\Rightarrow \begin{cases} m\dot{v}_x = -b v_x \\ m\dot{v}_y = -mg - b v_y \end{cases}$$



Chose the the upward direction as the positive direction

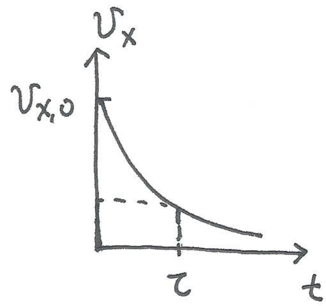
In the x-direction

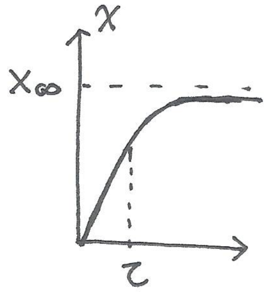
$$\frac{dv_x}{v_x} = -\frac{b}{m}dt \Rightarrow v_x(t) = A e^{-t/\tau}, \text{ where } \boxed{\tau = m/b} \leftarrow \text{time constant.}$$

Combine with $v_x(t=0) = v_0 \Rightarrow A = v_{x,0}$.

$$v_x = \frac{dx}{dt} \Rightarrow x(t) = x(0) + \int_0^t v_x\,dt = x(0) + \underbrace{\int_0^t}_{v_{x,0}} e^{-t'/\tau} dt'$$

Set $x(0) = 0 \Rightarrow x(t) = x_\infty (1 - e^{-t/\tau})$, where $x_\infty = v_{x,0}\tau$.





- Along the y-direction, it's an inhomogeneous 1st order ODE

$\dot{v}_y = -g - b/m\, v_y$. It's solution at $t \to \infty$, $\dot{v}_y = 0 \Rightarrow v_y(t \to \infty) = mg/b$ ✓
$\triangleq -v(\infty)$

$$\Rightarrow v_y(t) = A e^{-t/\tau} - v(\infty) \leftarrow \text{a special solution}$$

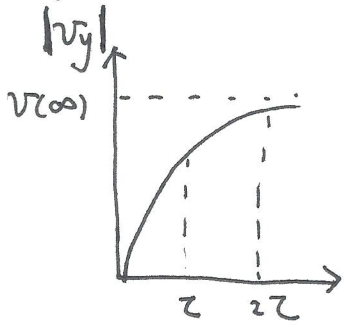
solution to the homogeneous part

plug in $v_y(t=0) = v_{y,0} \Rightarrow A = v_{y,0} + v(\infty)$

$$\boxed{\Rightarrow v_y(t) = v_{y,0}\, e^{-t/\tau} - v(\infty)(1 - e^{-t/\tau})}$$

$$y(t) = \int_0^t v_y(t')dt' = -v(\infty)t + (v_{y0} \boxed{+} v(\infty))\tau(1 - e^{-t/\tau}) \quad (\text{set } y(t=0)=0)$$

If $v_y(t=0) = 0$, we have $v_y = -v(\infty)(1 - e^{-t/\tau})$



time	percent of $v(\infty)$
τ	63%
2τ	86%
3τ	95%

Estimation of orders

① $v(\infty) = \frac{mg}{b} = \frac{\rho \pi D^3 g}{6 D 3\pi \eta} = \frac{\rho D^2 g}{18 \eta}$

For an oil drop in Millikan experiment, $D = 1.5\,\mu m$. $\rho = 840\,Kg/m^3$

$\Rightarrow v(\infty) = 6.1\times10^{-5}\,m/s$.

But for size $D = 0.2\,mm$, $\Rightarrow v(\infty) \simeq 1.3\,m/s$.
of drizzle drop

② time scale : $\tau = m/b = \frac{v(\infty)}{g}$

For millikan drop $\Rightarrow \tau \approx 6\times10^{-6}\,s$

For drizzle drop $\rightarrow \tau \approx 0.13\,s$.

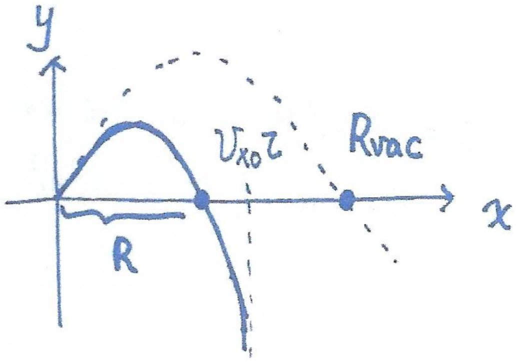
§ Trajectory and range

$$\left.\begin{aligned} &① \; x(t) = v_{x0}\tau(1-e^{-t/\tau}) \\ &② \; y(t) = (v_{y0}+v(\infty))\tau(1-e^{-t/\tau}) - v(\infty)t \end{aligned}\right\} \text{ with } x(0)=y(0)=0.$$

Form ① $\Rightarrow t = -\tau \ln\left(1-\frac{x}{v_{x0}\tau}\right)$

$$\boxed{y = \frac{v_{y0}+v_\infty}{v_{x0}}\,x + v(\infty)\tau\ln\left(1-\frac{x}{v_{x0}\tau}\right)}$$

x cannot exceed $x_\infty = v_{x0}\tau$.



The range on the horizontal direction in the vacuum

$$R_{vac} = \frac{2 v_{x0} v_{y0}}{g}.$$

Now with resistance, we solve $\frac{v_{y0}+v(\infty)}{v_{x0}} R + v(\infty)\tau \ln\left(1-\frac{R}{v_{x0}\tau}\right) = 0$

In the limit of small resistance, τ is large, that $\frac{R}{v(\infty)\tau} \ll 1$

we use $\ln(1-\epsilon) = -(\epsilon + \frac{1}{2}\epsilon^2 + \frac{1}{3}\epsilon^3 + \cdots)$

$$\Rightarrow \frac{v_{y0}+v(\infty)}{v_{x0}} R - v(\infty)\tau \left[\frac{R}{v_{x0}\tau} + \frac{1}{2}\left(\frac{R}{v_{x0}\tau}\right)^2 + \frac{1}{3}\left(\frac{R}{v_{x0}\tau}\right)^3\right] \simeq 0$$

$$\Rightarrow \frac{v_{y0}}{v_{x0}} \frac{R}{v(\infty)\tau} = \frac{1}{2}\left(\frac{R}{v_{x0}\tau}\right)^2 + \frac{1}{3}\left(\frac{R}{v_{x0}\tau}\right)^3$$

$$\Rightarrow R = \frac{2 v_{x0} v_{y0}}{g} - \frac{2}{3 v_{x0}\tau} R^2$$

$$\Rightarrow \boxed{R \approx R_{vac} - \frac{2}{3 v_{x0}\tau} \frac{4 v_{x0}^2 v_{y0}^2}{g^2} = R_{vac}\left[1 - \frac{4}{3}\frac{v_{y0}}{v(\infty)}\right]}$$

when $\frac{v_{y0}}{v(\infty)} \simeq 1$, the effect of air-resistance cannot be neglected!

Estimations: A metal pellets $D = 0.2\,mm$, $v = \phi\ 1\,m/s$ at angle 45°.

The range in the absence of resistance: $R_{vac} = \frac{2 v_{x0} v_{y0}}{g} = \frac{v^2 \sin 2\theta}{g} = \frac{1}{9.8}$

$\approx 10.2\,cm$

For gold $v(\infty) = \frac{\rho D^2 g}{18\eta}$ plug in $\rho = 16\ g/cm^3$

$D = 0.2mm.$

$\eta = 1.7 \times 10^{-5}\ kg/m\cdot s$

$\Rightarrow v(\infty) = 21 m/s$

the correction $\frac{4}{3} \cdot \frac{1\times 0.7}{21} \simeq 5\%$

For Al, $\rho = 2.7\ g/cm^3$, which is $1/6$ of gold.

hence, the correction is 6 times larger. $\simeq 30\%$.