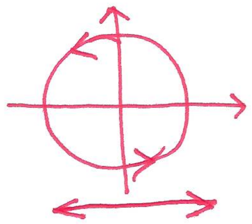
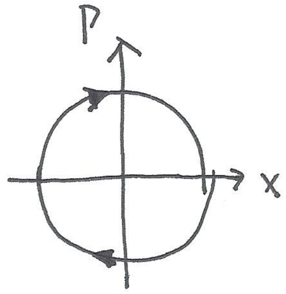
Lect 5: Newton's laws of motion (2) — oscillation

Outline:

1. Harmonic oscillator and uniform circular motion

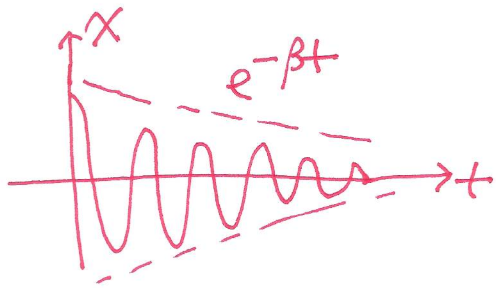


2. phase space orbit, Bohr-Sommerfeld condition



$$\oint p\,dx = (n+1/2)\,h$$

$$\frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 = \frac{1}{2}m\omega^2 A^2$$



3. Damped harmonic oscillators

$$\ddot{x} + \frac{1}{\tau}\dot{x} + \omega_0^2 x = 0$$

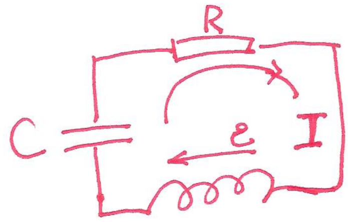
— overdamp
— underdamp
— critical

$$x(t) = A\cos(\omega t + \phi)\,e^{-\beta t}$$

$$\omega = \omega_0\sqrt{1-\left(\frac{1}{2\omega_0\tau}\right)^2}, \quad \beta = \frac{1}{2\tau}$$

4: Cyclotron orbit: 

$$R = \sqrt{\frac{\hbar c}{qB}}$$



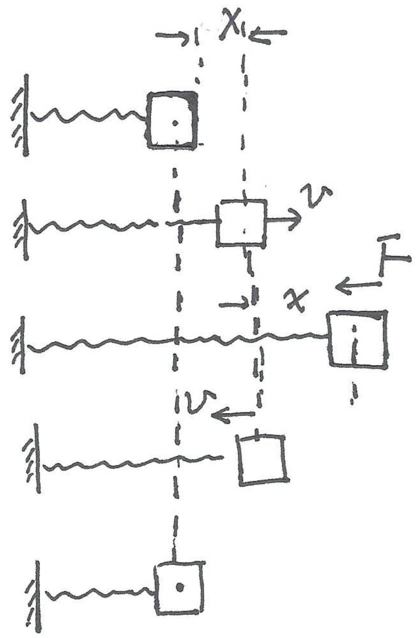
5: LC oscillators

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = 0.$$

§ Harmonic oscillators

Oscillations are a general class of phenomena, such as the elastic spring oscillator, simple pendulum, and the electromagnetic analogy of LC oscillator, etc. Quantum mechnically, each quantum mode in the free case can be viewed as an harmonic oscillator, such as the photo mode, the lattice vibration — phonon modes, etc. Waves can be viewed as a series of vibration modes propagating in space-time, including mechanical wave, E&M wave, and quantum mechanical, gravitational waves.

Harmonic oscillator is a proto-type system to study in all branches of physics. It is simple, elegent, both in a classic way and in the quantum way. Harmonic oscillator is also the first QM problem solved by Heisenberg.



According to Hook's law, the restoring force

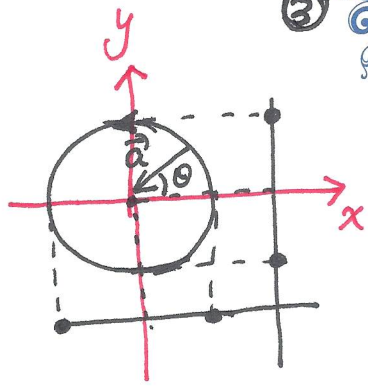
$F = -k x$, where x is measured from its equilibrium position

Then

$$\boxed{m\frac{d^2x}{dt^2} = -kx}$$

This a 2nd order constant coefficient differential equation. Actually, we could find an elementary but smart method to solve it!

Consider a uniform circular motion, with the radius A and circular frequency ω. The acceleration



$$\vec{a} = -a\,\hat{e}_r = -a(\cos\theta\,\hat{x} + \sin\theta\,\hat{y})$$
$$= -\frac{a}{A}(x\hat{x} + y\hat{y})$$
$$= -\omega^2(x\hat{x} + y\hat{y})$$

let's project the motion to the x-direction, we have

$$a_x = \frac{d^2x}{dt^2} = -\omega^2 x,$$

which is the same as the harmonic oscillator's equation with the identification $\omega = \sqrt{k/m}$.

$\theta = \omega t$ is obvious. Hence the solution to the uniform circular motion is

$$\begin{cases} x = A\cos(\omega t + \varphi) \\ y = A\sin(\omega t + \varphi), \end{cases}$$

where A and φ are integral consts to be determined by the initial conditions.

$$\boxed{\begin{array}{l} x(0) = A\cos\varphi \\ \dot{x}(0) = -\omega A\sin\varphi \end{array} \Rightarrow \begin{cases} A^2 = x^2(0) + (\dot{x}(0)/\omega)^2 \\ \tan\varphi = -\dfrac{\dot{x}(0)}{\omega x(0)} \end{cases}}$$

Generally speaking, this kind of differential equation can be solved by trying $x(t) = e^{\lambda t}$, plug in $\ddot{x} = -\omega^2 x \Rightarrow \lambda^2 = -\omega^2 \Rightarrow \lambda = \pm\omega i$

Hence $x(t) = a_1 e^{i\omega t} + a_2 e^{-i\omega t}$. If we require $x(t)$ is real, then

$a_1 = a_2^* = Ae^{i\varphi}$, then $\Rightarrow$ $\boxed{x(t) = A\cos(\omega t + \varphi)}$

$$\boxed{v(t) = \dot{x}(t) = -A\omega\sin(\omega t + \varphi)}$$

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We have seen the relation between the uniform circular motion and the harmonic oscillation. Actually the relation is even closer, if we oragnize

$$\boxed{-v(t)/A\omega = \sin(\omega t+\varphi), = y(t),}$$

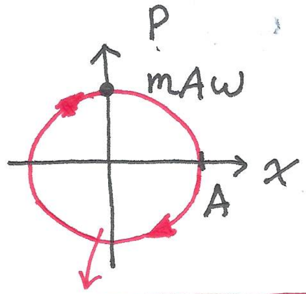
hence, the y-axis motion in fact reflects the velocity of the oscillation!

㊀ Phase space orbits

In the canonical version of classic mechanics, momentum $\vec{p}=m\vec{v}$ is also a fundament quantity. Based on the above reasoning,

$$\begin{cases} x = A\cos(\omega t+\varphi) \\ p = mv = -mA\omega\sin(\omega t+\varphi) \end{cases}$$

$$\Rightarrow \left(\frac{x}{A}\right)^2 + \left(\frac{p}{mA\omega}\right)^2 = 1$$



Circular (elliptical) orbit in the phase space

Please note that the phase space orbit is chiral, i.e., it only rotates in one direction, but not in the reverse direction! — c.f. the cyclotron motion of electrons in the magnetic field

Classically, the area enclosed by the orbit is arbitary. Quantum mechanically, however, the area has a minimum $\frac{h}{2}$.

$$\boxed{\oint p\,dx = (n+\tfrac{1}{2})\,h, \quad n=0,1,2,\cdots 3 \quad \text{over a period}}$$

Bohr-Sommerfeld quantization condition

minimal area

$$\pi \cdot A_{min}^2\, m\,\omega = \frac{h}{2}$$

$$\Rightarrow l_0 = A_{min} = \sqrt{\frac{\hbar}{m\omega}}$$

Hence, for a harmonic oscillation, there exists a minimum length scale $l_0 = \sqrt{\frac{\hbar}{m\omega}}$.

uncertainty principle: Since the orbit area has a minimum, it means oscillators cannot be at rest. The uncertainty $\sqrt{\langle \Delta x^2 \rangle}\sqrt{\langle (\Delta p)^2 \rangle}$ is at the same order of the area of the minimum orbit. Roughly speaking

$$\sqrt{\langle (\Delta x)^2 \rangle}\sqrt{\langle (\Delta p)^2 \rangle} \sim \hbar$$

$\leftarrow$ a more precise calculation based on quantum mechanics shows

$$\sqrt{\langle (\Delta x)^2 \rangle}\sqrt{\langle (\Delta p)^2 \rangle} \geq \frac{\hbar}{2}.$$

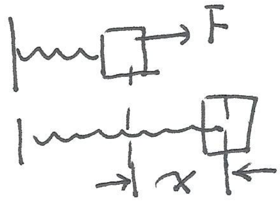
(*) Conservation law:

we have seen $\left(\frac{x}{A}\right)^2 + \left(\frac{p}{mA\omega}\right)^2 = 1 \Rightarrow \boxed{\frac{p^2}{2m} + \frac{1}{2} m \omega^2 x^2 = \frac{1}{2} m \omega^2 A^2}$

The first term is the kinetic energy: $p^2/2m = \frac{1}{2} m v^2$.

The 2nd term is determined by the status of the spring. Let's calculate the work done to change the spring length to x

$$W = \int_0^x F\, dx' = K \int_0^x x'\, dx' = \frac{1}{2} K x^2 = \frac{1}{2} m \omega^2 x^2$$



$\Rightarrow$ $\boxed{\text{kinetic energy} + \text{potential energy} = \text{total energy}}$

⊛ Damped harmonic oscillators

Consider a harmonic oscillator but with a ~~linear~~ linear resistance.

$$F = m\ddot{x} = -m\omega_0^2 x - b\dot{x} \quad \Rightarrow \quad \boxed{\ddot{x} + \omega_0^2 x + \frac{1}{\tau}\dot{x} = 0}$$

where $\tau = m/b$.

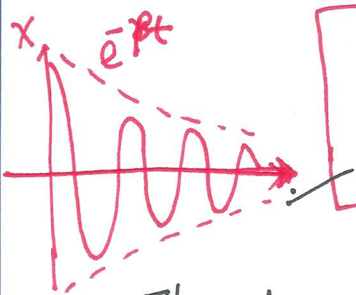
trying solution: ① $x(t) \sim e^{i\lambda t} \Rightarrow -\lambda^2 + \frac{i\lambda}{\tau} + \omega_0^2 = 0$

$$\lambda_{1,2} = \pm\sqrt{\omega_0^2 - \left(\frac{1}{2\tau}\right)^2} + \frac{i}{2\tau}$$

friction decreas frequency

hence, $x(t) = \left(a_1 e^{i\omega t} + a_2 e^{-i\omega t}\right) e^{-\frac{t}{2\tau}}$, where $\omega = \sqrt{\omega_0^2 - \left(\frac{1}{2\tau}\right)^2}$

Consider the real solution $\Rightarrow$



$$\boxed{x(t) = A\cos(\omega t + \phi)\, e^{-\beta t}, \text{ with } \omega = \omega_0\sqrt{1 - \left(\frac{1}{2\omega_0\tau}\right)^2}, \quad \beta = \frac{1}{2\tau}}$$

The above solution works at $\frac{1}{2\tau} < \omega_0$, i.e. $\omega_0\tau > 1/2$, This situation is called the underdamped case.

Suppose we know the initial condition x_0, and v_0, how to determine a_1, a_2?

$x_0 = a_1 + a_1^*$ $\leftarrow$ choose $a_2 = a_1^*$

$$\frac{dx}{dt} = \left(i a_1 \omega e^{i\omega t} - i a_1^* \omega e^{-i\omega t}\right) e^{-t/2\tau} + \left(\frac{-1}{2\tau}\right)\left(e^{-t/2\tau}\right) x(t)$$

$$v_0 = i\omega(a_1 - a_1^*) - \frac{1}{2\tau}(a_1 + a_1^*)$$

$$\Rightarrow \begin{cases} x_0 = 2a_R \\ v_0 = -2a_I\omega - \frac{1}{\tau}a_R \end{cases} \quad \Rightarrow \quad a_R = \frac{x_0}{2}, \quad a_I = \frac{v_0 + \frac{1}{\tau}\frac{x_0}{2}}{2\omega}$$

$$\Rightarrow \boxed{x = e^{-t/2\tau}\left[x_0\cos\omega t + \frac{v_0 + x_0/2\tau}{\omega}\sin\omega t\right]}$$

Quality factor

$$\boxed{Q = \omega\tau \approx \omega_0\tau}$$

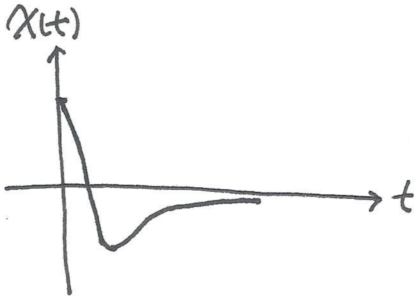
how many turns of oscillations before it half-decays.

② One the other hand, if $\omega_0\tau < 1/2$, there will be transient solutions. Try $x(t) \sim e^{-\lambda t}$, $\lambda^2 - \lambda/\tau + \omega_0^2 = 0$

$$\lambda_{1,2} = \frac{1}{2\tau} \pm \sqrt{\left(\frac{1}{2\tau}\right)^2 - \omega_0^2}$$

$$x(t) = a_1 e^{-\lambda_1 t} + a_2 e^{-\lambda_2 t}$$

plug in the initial conditions x_0 and v_0, $\Rightarrow$ $\begin{cases} a_1 = \dfrac{\lambda_1 x_0 + v_0}{\lambda_1 - \lambda_2} \\ a_2 = -\dfrac{\lambda_2 x_0 + v_0}{\lambda_1 - \lambda_2} \end{cases}$



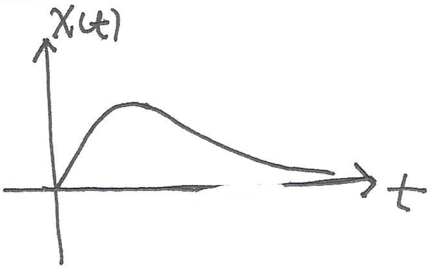
③ critical case $\omega_0\tau = 1/2$, $\lambda = \frac{1}{2\tau}$

$$x(t) = a_1 e^{-\lambda t} + a_2 t e^{-\lambda t}$$

$$\begin{cases} x_0 = a_1 \\ v_0 = -\lambda a_1 + a_2 \end{cases} \quad a_2 = v_0 + x_0/2\tau$$

$$\Rightarrow \boxed{x(t) = x_0 e^{-\frac{t}{2\tau}} + \left(v_0 + \frac{x_0}{2\tau}\right) t e^{-\frac{t}{2\tau}}}$$

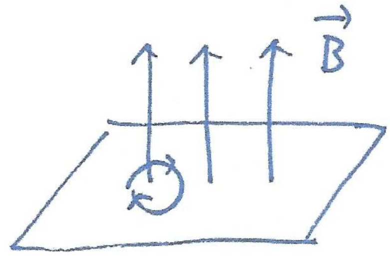
if $x_0 = 0$.



← decays quickest!

Other examples of oscillations

§ Motion in a uniform magnetic field



$$\boxed{\vec{F} = \frac{q}{c}\,\vec{v} \times \vec{B}}\,.$$ Set $\vec{B} = B\hat{z}$.

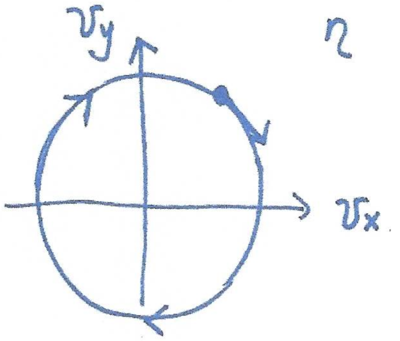
↖ Gaussian Unit

$$\begin{cases} m\dot{v}_x = \frac{q}{c} B\, v_y \\ m\dot{v}_y = -\frac{qB}{c} v_x \\ m\dot{v}_z = 0 \end{cases} \rightarrow \quad \omega = \frac{qB}{mc} \quad \text{cyclotron frequency}$$

$$\Rightarrow \begin{cases} \dot{v}_x = \omega v_y \\ \dot{v}_y = -\omega \, v_x \end{cases} \Rightarrow v_x + i v_y = -i\omega(\dot{v}_x + i\dot{v}_y)$$

Define $\eta = v_x + i v_y$ $\Rightarrow$ $\boxed{\eta = -i\omega\dot{\eta}}$

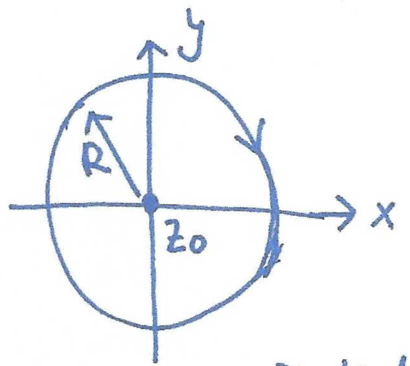
$\Rightarrow \eta = A e^{-i\omega t}$ with $A = v_x(0) + i v_y(0)$



we also define $z = x + iy$

$$\Rightarrow z = \text{const} + \int^t \eta \, dt = z_0 + \frac{iA}{\omega} e^{-i\omega t}$$

↑ center of the circle.



The cyclotron radius $R = \left|\frac{A}{\omega}\right| = \frac{vmc}{qB}$.

$\Rightarrow$ momentum magnitude $P = \frac{qBR}{c}$.

→ orbial angular momentum $L = PR = \frac{qBR^2}{c}$

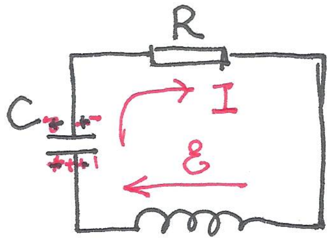
Classically, the circular orbital can be at any size.

But quantum mechanically, it has a minimal size.

The orbital angular momentum $L_{min} = \frac{qBR^2}{c} = \hbar$

$$\Rightarrow R = \sqrt{\frac{\hbar c}{qB}}$$ ← the cyclotron radius.

§ LC oscillators



$$\mathcal{E} = -L\frac{dI}{dt} = IR + \frac{Q}{C}$$

$\mathcal{E} = -L\frac{dI}{dt}$

$$\Rightarrow L\frac{dI}{dt} + IR + \frac{Q}{C} = 0$$

$$\frac{d^2Q}{dt^2} + \frac{R}{L}\frac{dQ}{dt} + \frac{Q}{LC} = 0$$

mapping $Q \to x$, $I \to \dot{x}$

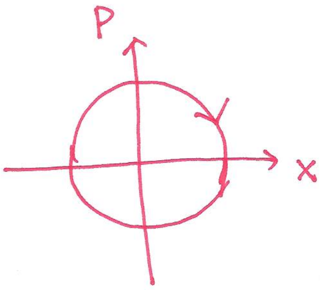
$$\begin{cases} L \to m, \quad \frac{1}{C} = K \\ \frac{R}{L} = \frac{1}{\tau} \Rightarrow \boxed{\omega_0^2 = K/m = \frac{1}{LC}} \end{cases}$$

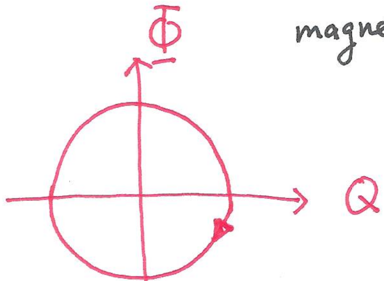
The analogy to momentum is

$$\boxed{\begin{array}{ccc} p = mv & \longrightarrow & LI = \Phi \\ x & \longrightarrow & Q \end{array}}$$

$$\left(\frac{x}{A}\right)^2 + \left(\frac{p}{m\omega A}\right)^2 = 1$$

$$\left(\frac{Q}{Q_0}\right)^2 + \left(\frac{\Phi}{LQ_0\omega_0}\right)^2 = 1$$





magnetic energy

⇅

electric energy.