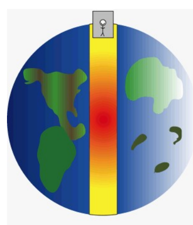
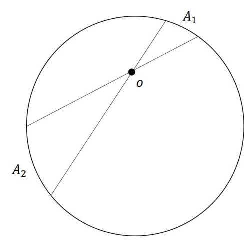
Problem one: The Longest Fall

In Liu Cixing's scientific fiction "The earth cannon (the longest fall)", a new method to travel between any two points on the earth is described. Consider two endpoints of a diameter of the earth, which are farthest from each other. You build a tunnel running through the interior of the earth, and just jump into it and fall to the other end as shown in Fig 1 (a). You do not need to worry about the internal structure of the earth and assume that the density of the earth is uniform.



(a) Gravity elevator



(b) Spherical shell

Figure 1: The longest fall

1. We can firstly study the gravity inside a spherical shell. Please make the argument that, there is no gravity inside the spherical shell.
   (Hint: Consider the situation in Figure (b). For an arbitrary point inside the spherical shell, verify that the forces generated by A_1 and A_2 cancel each other.)

2. For a point outside the spherical shell, you can treat the gravity as if all of its mass are concentrated in the center of the shell. This is a highly non-trivial statement that Newton spent many years to prove it. You do not need to prove it here, and just take it as a fact.

   Then please prove that you will do a harmonic oscillation after you jump into the tunnel. What is the period of the motion?

3. Consider a man-made satellite skimming the surface of earth. Please verify that your motion is just the projection of the circular motion of the satellite to the tunnel.

4. To be more practical, we would like to build the tunnel (still straight) to connect two general points on the earth, i.e., the tunnel does not need to pass the earth center.

In this case, you may sit in an elevator in this tunnel. The elevator has no power and its motion is confined along the tunnel. Please describe the motion of this gravity elevator and calculate its period. (For simplicity, you can regard the motion is frictionless)

Problem 2: Four springs harmonic oscillation

Consider the mass attached to four identical springs, as shown in Figure. Each spring has force constant k and unstretched length l_0, and the length of each spring when the mass is at its equilibrium at the origin is a (not necessarily the same as l_0). When the mass is displaced a small distance to the point (x, y), show that its potential energy has the form $\frac{1}{2}k'r^2$ appropriate to an isotropic harmonic oscillator. What is the constant k' in terms of k? Give an expression for the corresponding force.

Problem 3: Driven harmonic oscillation

A spar buoy of uniform cross-section floats in a vertical position with a length L submerged when there are no waves on the ocean. Please describe the motion of the spar buoy when there are sinusoidal waves of height h (crest to trough) and the period T on the ocean.

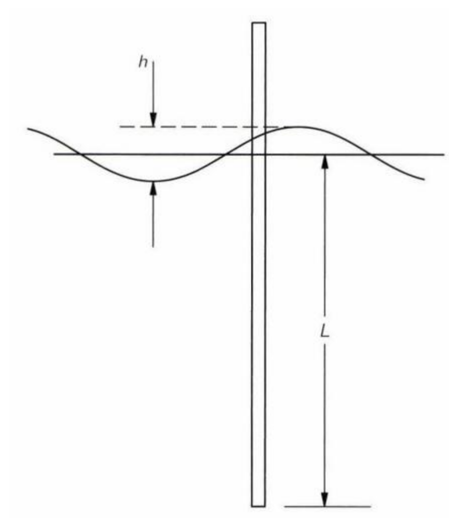


Figure 2: The spar buoy in the ocean

Problem 4: Damped harmonic oscillation I

A mass m is suspended from a spring of force constant k in a medium which exerts a damping form $-m\gamma dx/dt$.

1. For the case of under damped motion find the complete solutions for the position $x = x(t)$ of m for all times $t > 0$ for the following driving forces:

   (a)
   $$F = \begin{cases} 0 & \text{for } t < 0 \\ F_0 & \text{for } t \geq 0 \end{cases} \tag{1}$$

   (b)
   $$F = \begin{cases} 0 & \text{for } t < 0 \\ F_0 \cos \omega_0 t & \text{for } t \geq 0 \end{cases} \tag{2}$$
   $$\omega_0 = \sqrt{k/m}$$

2. If the oscillator is driven by a sinusoidal force $F = F_0 \cos \omega t$ and we consider long times, what is the frequency ω^* for which the amplitude reaches a maximum?

Problem 5: Damped oscillation II

Consider a damped oscillator, with natural frequency ω_0 and damping constant β both fixed, that is driven by a force $F(t) = F_0 \cos \omega t$.

1. Find the rate $P(t)$ at which $F(t)$ does work and show that the average rate $\langle P \rangle$ over any number of complete cycles is $m\beta\omega^2 A^2$.
2. Verify that this is the same as the average rate at which energy is lost to the resistive force.
3. Show that as w is varied $\langle P \rangle$ is maximum when $\omega = \omega_0$; that is, the resonance of the power occurs at $\omega = \omega_0$ (exactly).