Lecture 10 Motion in a Schwarzschild background

- $$S = -\int d\tau \; g_{\mu\nu} \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}$$

$$\mathcal{L} = \left(1 - \frac{2G_N M}{c^2 r}\right)\left(\frac{c\,dt}{d\tau}\right)^2 - \left(1 - \frac{2G_N M}{c^2 r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 - r^2\left(\frac{d\theta}{d\tau}\right)^2 - r^2 \sin^2\theta \left(\frac{d\phi}{d\tau}\right)^2$$

- $$\frac{d^2u}{d\phi^2} = \frac{G_N M}{\ell^2} - u + \frac{3G_N M u^2}{c^2} \qquad u = 1/r$$

- light $d\tau^2 = 0$ $\quad g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0$

$$\Rightarrow \frac{d^2u}{d\phi^2} = -u + \frac{3G_N M u^2}{c^2}$$

$$\boxed{\begin{aligned} &\cdot \text{ conservation: } \quad r^2\dot{\phi} = \ell \\ &\left(1 - \frac{2G_N M}{c^2 r}\right)\frac{dt}{d\tau} = k \end{aligned}}$$

$$\Delta\phi = \frac{3\pi}{2} r_s \left(\frac{1}{r_1} + \frac{1}{r_2}\right) \rightarrow \Delta\phi/100\text{ years} \simeq 43''$$

bending of light $\dfrac{2r_s}{r_0} \simeq 1.75''$

$r_s \simeq 3\text{km}, \; r_0 \simeq 7\times10^5\text{km}$

Action for massive / massless particles

Previously, we use $S = -m \int \sqrt{-g_{\mu\nu}(x)\, dx^\mu dx^\nu} \doteq -m \int d\tau$

It can be written $S = -m \int d\tau \frac{d\tau^2}{d\tau^2} = -m \int d\tau \; g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}$.

But for massless paricles, $d\tau = 0$, we cannot use this.

Consider $S[x^\mu, \sigma] = -\frac{1}{2} \int d\lambda \; \sigma \left(-g_{\mu\nu}(x) \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}\right) + \frac{m^2}{\sigma}$

$\delta S / \delta \sigma = 0 \Rightarrow -g_{\mu\nu}(x) \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} - \frac{m^2}{\sigma^2} = 0$

$$\boxed{\sigma = m \Big/ \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} \text{ if } m \neq 0; \text{ or } \Rightarrow g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = \text{ for } m = 0.}$$

variation to x^μ $\Rightarrow \frac{d}{d\lambda}\left(\sigma g_{\mu\nu}(x) \frac{dx^\nu}{d\lambda}\right) = \frac{\sigma}{2} \partial_\mu (g_{\nu\delta}) \frac{dx^\nu}{d\lambda} \frac{dx^\delta}{d\lambda}$

$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial (dx^\mu/d\lambda)}\right) = \frac{\partial L}{\partial x^\mu}$

define the affine parameter

$d\lambda = \sigma(\lambda)\, d\lambda'$

$\rightarrow \frac{1}{d\lambda} = \frac{1}{\sigma(\lambda)} \frac{d}{d\lambda'}$

$\frac{1}{\sigma(\lambda)} \frac{d}{d\lambda'}\left(g_{\mu\nu}(x) \frac{d}{d\lambda'} x^\nu\right)$

$= \frac{1}{2} \partial_\mu (g_{\nu\delta}) \frac{dx^\nu}{d\lambda'} \frac{dx^\delta}{d\lambda'} \frac{\sigma(\lambda)}{\sigma^2(\lambda)}$

$dx' = d\lambda \sqrt{-g_{\mu\nu}}$

$= d\tau$.

$\Rightarrow \frac{d}{d\lambda'}\left(g_{\mu\nu}(x) \frac{d}{d\lambda'} x^\nu\right) = \frac{1}{2} \partial_\mu g_{\nu\delta} \frac{dx^\nu}{d\lambda'} \frac{dx^\delta}{d\lambda'}$

remove the prime → $d\lambda' = d\tau$

$$\frac{d}{d\tau}\left[g_{\mu\nu}(x)\frac{d}{d\tau}x^{\nu}\right] - \frac{1}{2}\partial_\mu g_{\nu\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau} = 0$$

$$g_{\mu\nu}\frac{d^2}{d\tau^2}x^\nu + \partial_\sigma g_{\mu\nu}\frac{dx^\sigma}{d\tau}\frac{dx^\nu}{d\tau} - 1/2\,\partial_\mu g_{\nu\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau} = 0$$

$$g_{\mu\nu}\frac{d^2}{d\tau^2}x^\nu + 1/2\left(\partial_\sigma g_{\mu\nu} + \partial_\nu g_{\mu\sigma} - \partial_\mu g_{\nu\sigma}\right)\frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau} = 0$$

$$\frac{d^2}{d\tau^2}x^\lambda + 1/2\, g^{\lambda\mu}\left(\partial_\sigma g_{\mu\nu} + \partial_\nu g_{\mu\sigma} - \partial_\mu g_{\nu\sigma}\right)\frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau} = 0$$

← 采用 $d\tau$, we have

$$\boxed{\frac{d^2}{d\tau^2}x^\lambda + \Gamma^\lambda_{\sigma\nu}\frac{dx^\sigma}{d\tau}\frac{dx^\nu}{d\tau} = 0} \leftarrow \boxed{S = -m\int d\tau\, g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}$$

② If $m = 0$ → $g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0$ constraint

$$S = -1/2\int d\lambda\, \sigma\left(-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\right)$$ variation with respect to σ

Define $d\lambda = \sigma(\lambda)\, d\lambda' \Rightarrow \frac{1}{d\lambda} = \frac{1}{\sigma(\lambda)\, d\lambda'}$

$$S = \int d\lambda'\, \sigma^2(\lambda)/\sigma^2(\lambda)\; g_{\mu\nu}\frac{dx^\mu}{d\lambda'}\frac{dx^\nu}{d\lambda'}$$

$$= \int d\lambda'\; g_{\mu\nu}\frac{dx^\mu}{d\lambda'}\frac{dx^\nu}{d\lambda'} \rightarrow \boxed{\frac{d^2}{d\lambda'^2}x^\mu + \Gamma^\mu_{\nu\sigma}\frac{dx^\nu}{d\lambda'}\frac{dx^\sigma}{d\lambda'} = 0}$$

λ': affine parameter

§ Motion of a particle in a Schwarzschild background

$$c^2 d\tau^2 = \left(1 - \frac{2G_N M}{c^2 r}\right) c^2 dt^2 - \left(1 - \frac{2G_N M}{c^2 r}\right)^{-1} dr^2 - r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

$$S = \int d\tau = \int d\tau \; d\tau^2/d\tau^2 = -\int d\tau \; \underbrace{g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}}_{\mathcal{L}(t,\, r,\, \theta,\, \phi)}$$

$\mathcal{L}$ does not contain t, ϕ i.e. t, ϕ are cyclic variables.

$$\mathcal{L} = \left(1 - \frac{2G_N M}{c^2 r}\right) c^2 \left(\frac{dt}{d\tau}\right)^2 - \left(1 - \frac{2G_N M}{c^2 r}\right)^{-1} \left(\frac{dr}{d\tau}\right)^2 - r^2 \left(\frac{d\theta}{d\tau}\right)^2$$

$$- r^2 \sin^2\theta \left(d\phi/d\tau\right)^2 \qquad (*)$$

- $\frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot\theta}\right) = \frac{\partial \mathcal{L}}{\partial \theta}$ $\qquad \cdot \rightarrow \frac{d}{d\tau}$

$$\frac{d}{d\tau}\left(-2r^2 \frac{d\theta}{d\tau}\right) = -2r^2 \sin\theta \cos\theta \left(\frac{d\phi}{d\tau}\right)^2$$

$$-2\frac{d}{d\tau}\left(r^2 \dot\theta\right) + r^2 \sin 2\theta \left(\frac{d\phi}{d\tau}\right)^2 = 0$$

Consider the case of $\overbrace{\begin{cases} \theta = \pi/2, \\ \dot\theta = 0 \end{cases}}^{\text{initial condition}}$ then $\frac{d}{d\tau}\left(r^2 \dot\theta\right) = 0$, i.e.

$$r^2 \dot\theta = \text{const}$$

$\Rightarrow r^2 \dot\theta = 0$, i.e. $\dot\theta = 0$ for the motion.

$\rightarrow$ The motion lies in the equatorial plane

Then $\mathcal{L} = \left(1 - \frac{2G_N M}{c^2 r}\right) c^2 \dot t^2 - \left(1 - \frac{2G_N M}{c^2 r}\right)^{-1} \dot r^2 - r^2 \dot\phi^2$

②

$$\frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot\phi}\right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \Rightarrow \frac{d}{d\tau}\left(r^2 \sin^2\theta \, \dot\phi\right) = 0$$

$$r^2 \sin^2\theta \, \dot\phi = \text{const} = l .$$

if $\theta = \pi/2 \Rightarrow \boxed{r^2\dot\phi = l}$.

$$\frac{d}{d\tau}\frac{\partial \mathcal{L}}{\partial \dot t} = \frac{\partial \mathcal{L}}{\partial t} = 0 \Rightarrow \frac{d}{d\tau}\left[\left(1 - \frac{2G_N M}{\cancel{r}}\right)\dot t\right] = 0$$

$$\boxed{\left(1 - \frac{2G_N M}{c^2 r}\right)\frac{dt}{d\tau} = \text{const} = k}$$

$$\frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot r}\right) - \frac{\partial \mathcal{L}}{\partial r} = 0$$

$$\frac{d}{d\tau}\left(-2\left(1 - \frac{2G_N M}{c^2 r}\right)^{-1}\dot r\right) - \frac{2G_N M}{r^2}\dot t^2 - \left(1 - \frac{2G_N M}{c^2 r}\right)^{-2}\frac{2G_N M}{c^2 r^2}\dot r^2$$

$$+ 2r\dot\phi^2 = 0$$

$$-\left(1 - \frac{2G_N M}{c^2 r}\right)^{-1}\ddot r + \left(1 - \frac{2G_N M}{c^2 r}\right)^{-2}\frac{2G_N M}{c^2 r^2}\dot r^2 - \frac{G_N M}{r^2}\dot t^2$$

$$-\left(1 - \frac{2G_N M}{c^2 r}\right)^{-2}\frac{G_N M}{c^2 r^2}\dot r^2 + r\dot\phi^2 = 0$$

$$\rightarrow \boxed{\left(1 - \frac{2G_N M}{c^2 r}\right)^{-1}\ddot r - \left(1 - \frac{2G_N M}{c^2 r}\right)^{-2}\frac{G_N M}{c^2 r^2}\dot r^2 + \frac{G_N M}{r^2}\dot t^2 - r\dot\phi^2 = 0}$$

Based on the line element, and set $\theta = \pi/2$

$$\Rightarrow c^2 = \left(1 - \frac{2G_N M}{c^2 r}\right)c^2\dot t^2 + \left(1 - \frac{2G_N M}{c^2 r}\right)^{-1}\dot r^2 - r^2\dot\phi^2$$

$$\frac{c^2}{\dot\phi^2} = \left(1 - \frac{2G_N M}{c^2 r}\right)\frac{c^2\dot t^2}{\dot\phi^2} + \left(1 - \frac{2G_N M}{c^2 r}\right)^{-1}\dot r^2/\dot\phi^2 - r^2$$

$$\left(\frac{dr}{d\phi}\right)^2 - \left(1 - \frac{2G_N M}{c^2 r}\right)\frac{c^2 \dot{t}^2}{\dot{\phi}^2} + r^2\left(1 - \frac{2G_N M}{c^2 r}\right) = -\left(1 - \frac{2G_N M}{c^2 r}\right)\frac{c^2}{\dot{\phi}^2}$$

$$\left(\frac{dr}{d\phi}\right)^2 - \frac{k^2 r^4 c^2}{\ell^2} + r^2\left(1 - \frac{2G_N M}{c^2 r}\right) = -\left(1 - \frac{2G_N M}{r c^2}\right)\frac{c^2 r^4}{\ell^2}$$

$$\left(\frac{dr}{d\phi}\right)^2 = -r^2\left(1 - \frac{2G_N M}{c^2 r}\right) + \frac{r^4 c^2}{\ell^2}\left((k^2-1) + \frac{2G_N M}{c^2 r}\right)$$

Define $u = 1/r$, $\quad \frac{dr}{d\phi} = -\frac{1}{u^2}\frac{du}{d\phi}$

$$\Rightarrow \quad \frac{1}{u^4}\left(\frac{du}{d\phi}\right)^2 = -\frac{1}{u^2}\left(1 - \frac{2G_N M}{c^2}u\right) + \frac{c^2}{\ell^2 u^4}\left((k^2-1) + \frac{2G_N M u}{c^2}\right)$$

$$\left(\frac{du}{d\phi}\right)^2 = -u^2\left(1 - \frac{2G_N M}{c^2}u\right) + \frac{c^2}{\ell^2}\left((k^2-1) + 2\frac{G_N M}{c^2}u\right).$$

$$2\frac{du}{d\phi}\frac{d^2u}{d\phi^2} = \frac{6G_N M u^2}{c^2}\frac{du}{d\phi} + \frac{2G_N M}{\ell^2}\frac{du}{d\phi} - 2u\frac{du}{d\phi}$$

$$\left(\frac{d^2u}{d\phi^2} - \frac{G_N M}{\ell^2} + u - 3\frac{G_N M u^2}{c^2}\right)\frac{du}{d\phi} = 0$$

$\frac{du}{d\phi} = 0 \Rightarrow \frac{dr}{d\phi} = 0 \Rightarrow$ circular orbit

$$\boxed{\frac{d^2u}{d\phi^2} = \frac{G_N M}{\ell^2} - u + 3\frac{G_N M u^2}{c^2}}$$

GR correction!

- Newtonian Approximation:

- Based on $\mathcal{L} = (1-\frac{2G_NM}{c^2r})c^2\dot{t}^2 - (1-\frac{2G_NM}{c^2r})^{-1}\dot{r}^2 - r^2\dot{\phi}^2$

$c^2 \to \infty$. $dt/d\tau = 1$

$$\to \mathcal{L} = -\frac{2G_NM}{r} - \dot{r}^2 - r^2\dot{\phi}^2$$

$$\to -\frac{\mathcal{L}}{2} = \dot{r}^2 + r^2\dot{\phi}^2 + \frac{G_NM}{r} \longrightarrow \text{Newtonian mechanics}$$

$$\to \frac{d^2u}{d\phi^2} = \frac{G_NM}{\ell^2} - u = -(u - \frac{G_NM}{\ell^2})$$

$$u - \frac{G_NM}{\ell^2} = A\cos\phi \Rightarrow 1/r = A\cos\phi + \frac{G_NM}{\ell^2}$$

$$= \frac{G_NM}{\ell^2}[1 + e\cos\phi]$$

where $e = A\ell^2/G_NM$

$$\boxed{r = \frac{\ell^2/G_NM}{1 + e\cos\phi}}$$

elliptic orbit

- vertical fall $\dot{\phi} = 0 \to \ell = 0$

we directly start from

$$c^2 = (1-\frac{2G_NM}{c^2r})c^2\dot{t}^2 - (1-\frac{2G_NM}{r})^{-1}\dot{r}^2$$

$$\left(1-\frac{2G_NM}{c^2r}\right)\frac{dt}{d\tau} = k \Rightarrow c^2 = \frac{k^2c^2 - \dot{r}^2}{(1-\frac{2G_NM}{c^2r})}$$

$$\Rightarrow \dot{r}^2 - k^2c^2 + (1-\frac{2G_NM}{c^2r})c^2 = 0$$

If the particle fall from rest at $r = r_0$

$$\dot{r}\Big|_{r=r_0} = 0 \quad \Rightarrow \quad k = \left(1 - \frac{2G_N M}{c^2 r_0}\right)^{1/2}$$

$$\Rightarrow \quad \dot{r}^2 - c^2\left[1 - \frac{2G_N M}{c^2 r}\right] + c^2\left(1 - \frac{2G_N M}{c^2 r}\right) = 0$$

$$\frac{1}{2}\dot{r}^2 = G_N M\left(\frac{1}{r} - \frac{1}{r_0}\right)$$

$$\dot{r}\,\ddot{r} = -\frac{G_N M}{r^2}\,\dot{r} \quad \Rightarrow \quad \ddot{r} = -\frac{G_N M}{r^2}$$

However, r does not measure the radial distance

and $\dot{r} = dr/d\tau \leftarrow$ with respect to proper time.

$$\tau = \int d\tau = \int \frac{dr'}{\left[(2G_N M)\left(\frac{1}{r'} - \frac{1}{r_0}\right)\right]^{1/2}} = \frac{1}{\sqrt{2G_N M}} \int_r^{r_0} dr' \left(\frac{r_0 r'}{r_0 - r'}\right)^{1/2}$$

r can be extended to $r_s = \dfrac{2G_N M}{c^2}$, then the proper time experienced by the particle to r_s is finite.

But the coordiniate time

$$dt/d\tau = \frac{k}{\left(1 - \frac{2G_N M}{r}\right)} = \frac{\left(1 - \frac{2G_N M}{r_0}\right)^{1/2}}{1 - \frac{2G_N M}{r}}$$

$$t = \int d\tau \, \frac{\left(1 - \frac{2G_N M}{r_0}\right)^{1/2}}{1 - \frac{2G_N M}{r}} = \int_r^{r_0} \frac{dr'}{\sqrt{2G_N M}} \, \frac{\left(1 - \frac{2G_N M}{r_0 c^2}\right)^{1/2}}{\left(1 - \frac{2G_N M}{r c^2}\right)} \left(\frac{r_0 r'}{r_0 - r'}\right)^{1/2}$$

it diverges as $r' \to r_s = \dfrac{2G_N M}{c^2}$

$$v(r) = \frac{dr}{dt} = \frac{d\tau}{dt}\frac{dr}{d\tau} = (\dot{t})^{-1}\,\dot{r}$$

$$\frac{dr}{d\tau} = \dot{r} = \left[2G_N M\,\frac{r_0 - r}{r_0 r}\right]^{1/2}$$

$$v(r) = \frac{1 - \frac{2G_N M}{rc^2}}{\left(1 - \frac{2G_N M}{r_0 c^2}\right)^{1/2}}\left[2G_N M\,\frac{r_0 - r}{r_0 r}\right]^{1/2}$$

as $r \to r_s$, then $v(r) \to 0$.

~~Suppose the particle starts from $r_0 \to \infty$, $v(r) = 1 - \frac{2G_N M}{rc^2}$~~

* circular orbit $\quad dr/d\phi = 0 \Rightarrow \dot{r} = 0 = \ddot{r}$

plug in the radial Eq.

$$\frac{G_N M}{r^2}\left(\frac{dt}{d\tau}\right)^2 = r\dot{\phi}^2 \Rightarrow \frac{\dot{\phi}^2}{\dot{t}^2} = \left(\frac{d\phi}{dt}\right)^2 = \frac{G_N M}{r^3}$$

$$\int dt = \int\left(\frac{r^3}{G_N M}\right)^{1/2} d\phi = \left(\frac{r^3}{G_N M}\right)^{1/2}\cdot 2\pi$$

$$\Rightarrow \quad \Delta t = 2\pi\left(\frac{r^3}{G_N M}\right)^{1/2}$$

But r is not the true radius of the orbit

and Δt is the coordinate orbit period

An observer at rest at the orbit of r_0, the proper time of period he measures

$$\boxed{\Delta\tau_{space} = \sqrt{g_{00}(r_0)}\,\Delta t = \left(1 - \frac{2G_N M}{r}\right)^{1/2}\Delta t}$$

Suppose an observer moving with the planet, then the proper time

$$\left(1-\frac{2G_N M}{c^2 r}\right)\frac{\Delta t}{\Delta \tau} = k \Rightarrow \Delta\tau_{pl} = \left(1-\frac{2G_N M}{c^2 r}\right)k^{-1}\Delta t$$

$$c^2 = \left(1-\frac{2G_N M}{c^2 r}\right)c^2\dot{t}^2 - \left(1-\frac{2G_N M}{r}\right)^{-1}\dot{r}^2 - r^2\dot{\phi}^2$$

$$\dot{r}=0 \quad \Rightarrow \quad c^2 = \left(1-\frac{2G_N M}{c^2 r}\right)c^2\dot{t}^2 - r^2\dot{\phi}^2$$

radial Eq $\qquad \frac{G_N M}{r^2}\dot{t}^2 = r\dot{\phi}^2 \Rightarrow r^2\dot{\phi}^2 = \frac{G_N M}{r}\dot{t}^2$

$$c^2 = \left(1-\frac{3G_N M}{c^2 r}\right)c^2\dot{t}^2 \Rightarrow k = \left(1-\frac{2G_N M}{c^2 r}\right)\dot{t}$$

$$= \frac{1-\frac{2G_N M}{c^2 r}}{\sqrt{1-\frac{3G_N M}{c^2 r}}}$$

$$\Rightarrow \boxed{\Delta\tau_{pl} = \sqrt{1-\frac{3G_N M}{c^2 r}}\,\Delta t}$$

8 Motion of light ray

light behaves like other particle follows the geodesics

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} = 0$$

However, for photon $d\tau^2 = 0$, hence, we cannot use $d\tau$. We use another parameter along it world line λ

$\cdot \rightarrow d/d\lambda$

Similarly
$$r^2\dot\phi = l = const$$

$$\left(1-\frac{2G_N M}{c^2 r}\right)\dot t = k = const$$

$$\left(1-\frac{2G_N M}{c^2 r}\right)^{-1}\ddot r - \left(1-\frac{2G_N M}{c^2 r}\right)^{-2}\frac{G_N M \dot r^2}{c^2 r^2} + \frac{G_N M}{r^2}\dot t^2 - r\dot\phi^2 = 0$$

further:
$$d\tau^2 = 0 \Rightarrow g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0.$$
$$\Downarrow$$

$$\left(1-\frac{2G_N M}{c^2 r}\right)c^2\dot t^2 - \left(1-\frac{2G_N M}{c^2 r}\right)^{-1}\dot r^2 - r^2\dot\phi^2 = 0 \quad \swarrow \text{fix } \theta = \pi/2$$

$$\left(1-\frac{2G_N M}{c^2 r}\right)\frac{\dot t^2}{\dot\phi^2} - \left(1-\frac{2G_N M}{c^2 r}\right)^{-1}\left(\frac{dr}{d\phi}\right)^2 - r^2 = 0$$

$$\left(1-\frac{2G_N M}{c^2 r}\right)\frac{k^2}{\left(1-\frac{2G_N M}{c^2 r}\right)^2}\frac{r^4}{l^2} - \left(1-\frac{2G_N M}{c^2 r}\right)^{-1}\left(\frac{dr}{d\phi}\right)^2 - r^2 = 0$$

$$\boxed{\left(\frac{dr}{d\phi}\right)^2 + \left(1-\frac{2G_N M}{c^2 r}\right)r^2 - \frac{k^2 r^4}{l^2} = 0}$$

again $u = 1/r$ $\quad dr/d\phi = -\frac{1}{u^2}\, du/d\phi$

$$\frac{1}{u^4}\left(\frac{du}{d\phi}\right)^2 + \left(1 - \frac{2G_N M}{c^2}u\right)u^{-2} - \frac{k^2}{l^2}u^{-4} = 0$$

$$\left(\frac{du}{d\phi}\right)^2 + \left(1 - \frac{2G_N M}{c^2}u\right)u^2 - \frac{k^2}{l^2} = 0$$

$$\frac{du}{d\phi}\left(\frac{d^2u}{d\phi^2} + u - \frac{3G_N M}{c^2}u^2\right) = 0$$

The radial geodesic Eq $\quad \frac{du}{d\phi} = 0$, or $\frac{d^2u}{d\phi^2} + u - \frac{3G_N M}{c^2}u^2 = 0$

① $\frac{du}{d\phi} = 0 \Rightarrow \dot{r} = \ddot{r} = 0 \Rightarrow$

$$\frac{G_N M}{r^2}\dot{t}^2 = r\dot{\phi}^2$$

$$d\tau^2 = 0 \Rightarrow \left(1 - \frac{2G_N M}{c^2 r}\right)c^2\dot{t}^2 = r^2\dot{\phi}^2 = \frac{G_N M}{r}\dot{t}^2$$

$$\Rightarrow \left(1 - \frac{3G_N M}{c^2 r}\right)\dot{t}^2 = 0 \Rightarrow \boxed{r = \frac{3G_N M}{c^2}}$$

photon moves in a circular orbit

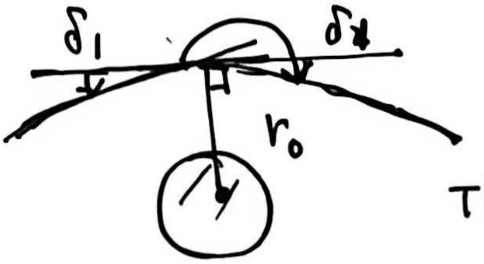
② Now consider

$$\frac{d^2u}{d\phi^2} + u - 3\frac{G_N M}{c^2}u^2 = 0$$

$$\frac{3G_N M}{c^2}\frac{u^2}{u} = \frac{3G_N M}{c^2 r} = \frac{3}{2}\, r_s/r \ll 1$$

$$\frac{d^2u}{d\phi^2} + u = 0 \leftarrow \text{first order}$$

$$u = 1/r = A \sin\phi \quad \Rightarrow \quad r\sin\phi = 1/A = \text{const} \doteq r_0$$



Then $$\frac{d^2u}{d\phi^2} + u = \epsilon u^2 \leftarrow \epsilon = 3\frac{G_N M}{c^2} = \frac{3}{2} r_s$$

Try $$u \simeq \frac{1}{r_0}\sin\phi + \epsilon u_1$$

$$\Rightarrow \cancel{\frac{-1}{r_0}\sin\phi} + \epsilon\frac{d^2u_1}{d\phi^2} + \cancel{\frac{1}{r_0}\sin\phi} + \epsilon u_1 = \epsilon\frac{1}{r_0^2}\sin^2\phi$$

$$d^2u_1/d\phi^2 + u_1 = \frac{1}{2r_0^2}(1-\cos 2\phi)$$

try $$u_1 = a + b\cos 2\phi \quad \Rightarrow \quad a = \frac{1}{2r_0^2} \quad b = \frac{1}{6r_0^2}$$

$$u(\phi) \simeq \frac{1}{r_0}\sin\phi + \epsilon\left(\frac{1}{2r_0^2} + \frac{1}{6r_0^2}\cos 2\phi\right) = 1/r$$

At $r \to \infty$, $1/r = 0$ $\Rightarrow$ $$\sin\phi + \epsilon\left[\frac{1}{2r_0}\right]\left(1 + \frac{\cos 2\phi}{3}\right] = 0$$

① $\phi \sim -\delta_1 \to 0$ ② $\phi \sim \pi + \delta_2 \to \pi$

$$-\delta_1 + \frac{\epsilon}{2r_0}\left(1+\frac{1}{3}\right) = 0 \quad \Rightarrow \quad \delta_1 = \frac{2\epsilon}{3r_0} = \frac{2G_N M}{r_0 c^2}$$

② $$-\delta_2 + \frac{\epsilon}{2r_0}(1+1/3) = 0 \quad \Rightarrow \quad \delta_2 = \frac{2G_N M}{r_0 c^2}$$

$\Rightarrow$ deflection of light ray $$\delta_1 + \delta_2 = \frac{4G_N M}{c^2 r_0} = \frac{2r_s}{r_0} \sim 1.75''$$

- Perihelim advance of Mercury

Kepler's problem $\quad d^2u/d\phi^2 = \dfrac{G_N M}{\ell^2} - u$

$$u(\phi) = \frac{1}{r} = \frac{G_N M}{\ell^2}\left[1 + e\cos\phi\right] \leftarrow \text{eliptic orbital}$$

eccentricity

$$E = \frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\dot{\phi}^2 - G_N M/r \qquad r^2\dot{\phi} = \ell$$

$$= \frac{1}{2}\left(\frac{d\phi}{dt}\frac{dr}{d\phi}\right)^2 + \frac{1}{2}\frac{\ell^2}{r^2} - \frac{G_N M}{r} \qquad \dot{\phi} = \ell/r^2$$

$$= \frac{\ell^2}{2}\left(\frac{du}{d\phi}\right)^2 + \frac{1}{2}\ell^2 u^2 - G_N M u$$

$$\left(\frac{du}{d\phi}\right)^2 = \frac{2E}{\ell^2} + \frac{2G_N M}{\ell^2}u - u^2$$

$$du/d\phi = -\frac{G_N M}{\ell^2}e\sin\phi \Rightarrow \left(\frac{GM}{\ell^2}e\sin\phi\right)^2 = \frac{2E}{\ell^2} + 2\left(\frac{G_N M}{\ell^2}\right)^2(1+e\cos\phi)$$

$$-\left(\frac{G_N M}{\ell^2}\right)^2(1+e\cos\phi)^2$$

$$\left(\frac{GM}{\ell^2}\right)^2\left[1 + \cancel{2e\cos\phi} + e^2\right] = \frac{2E}{\ell^2} + 2\left(\frac{G_N M}{\ell^2}\right)^2(1 + \cancel{e\cos\phi})$$

$$\left(\frac{G_N M}{\ell^2}\right)^2 e^2 = \frac{2E}{\ell^2} + \left(\frac{G_N M}{\ell^2}\right)^2$$

$$e^2 = 1 + \frac{2E\ell^2}{G_N^2 M^2}$$

aphelim 远日

perihelim 近日

$$r_{ap} = \frac{\ell^2}{G_N M}\frac{1}{1-e} = r_1, \quad u_{ap} = \frac{G_N M}{\ell^2}(1-e) = u_1$$

$$r_{pH} = \frac{\ell^2}{G_N M}\frac{1}{1+e} = r_2 \qquad u_{pH} = \frac{G_N M}{\ell^2}(1+e) = u_2$$

- Now consider the GR correction

Define $E = \frac{1}{2}\dot{r}^2 + \frac{1}{2} r^2 \dot{\phi}^2 - \frac{G_N M}{r} + \frac{G_N M \ell^2}{c^2 r^3}$

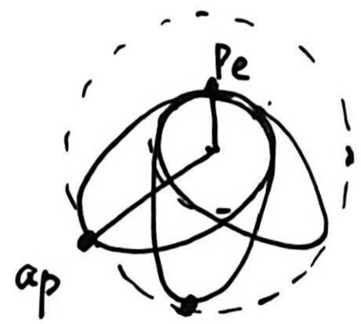
$$\rightarrow E = \frac{1}{2}\ell^2\left(\frac{du}{d\phi}\right)^2 + \frac{1}{2}\ell^2 u^2 - G_N M u + G_N M \ell^2 u^3/c^2$$

$$\left(\frac{du}{d\phi}\right)^2 = 2E/\ell^2 + \frac{2G_N M}{\ell^2} u - u^2 + \frac{2G_N M}{c^2} u^3$$

set $\epsilon = \frac{2G_N M}{c^2}$, which is a small parameter

$$u_{1,2}\,\epsilon \simeq \frac{2(G_N M)^2}{c^2 \ell^2} << 1$$

perihelion / aphelion



$$\epsilon u^3 - u^2 + \frac{2G_N M}{\ell^2} u + \frac{2E}{\ell^2} = 0$$

$$u^3 - 1/\epsilon\, u^2 + \frac{2G_N M}{\epsilon \ell^2} u + \frac{2E}{\epsilon \ell^2} = 0$$

$$u_1 \simeq \frac{G_N M}{\ell^2}(1-e)$$

$$u_2 \simeq \frac{G_N M}{\ell^2}(1+e)$$

$$u_3 \simeq 1/\epsilon - (u_1 + u_2)$$

$u_1 u_2 u_3 = \frac{G_N^2 M^2}{\epsilon \ell^4} \cdot \frac{2E\ell^2}{G_N^2 M^2} \simeq 2E/\epsilon\ell^2$

$u_1 u_2 + u_2 u_3 + u_3 u_1 \simeq u_3(u_1 + u_2) \sim 1/\epsilon \cdot \frac{2G_N M}{\ell^2}$

$$\Rightarrow \left(\frac{du}{d\phi}\right)^2 = \epsilon(u-u_1)(u_2-u)(u_3-u)$$

$$= \epsilon(u-u_1)(u_2-u)(1/\epsilon - u_1 - u_2 - u)$$

$$= (u-u_1)(u_2-u)[1-\epsilon(u_1+u_2+u)]$$

$$d\phi/du = \frac{[1-\epsilon(u+u_1+u_2)]^{-1/2}}{\{(u-u_1)(u_2-u)\}^{1/2}} \simeq \frac{1+1/2\,\epsilon(u+u_1+u_2)}{[(u-u_1)(u_2-u)]^{1/2}}$$

$$\alpha = \frac{u_1+u_2}{2} = G_N M/\ell^2$$

$$\beta = \frac{u_2-u_1}{2} = \frac{G_N M}{\ell^2}\cdot e$$

$$d\phi/du = \frac{1+\frac{1}{2}\epsilon u+\epsilon\alpha}{[\beta^2-(u-\alpha)^2]^{1/2}} = \frac{\frac{1}{2}\epsilon(u-\alpha)+1+\frac{3}{2}\alpha\epsilon}{[\beta^2-(u-\alpha)^2]^{1/2}}.$$

from $u_1 \to u_2$ (i.e aphelim $\to$ perihelim).

$$\delta\phi = \int_{u_1}^{u_2} du\, \frac{\frac{1}{2}\epsilon(u-\alpha)+1+\frac{3}{2}\alpha\epsilon}{\sqrt{\beta^2-(u-\alpha)^2}}$$

$$= \underbrace{-\frac{1}{2}\epsilon\,[\beta^2-(u-\alpha)^2]^{1/2}\Big|_{u_1}^{u_2}}_{0} + (1+\tfrac{3}{2}\epsilon\alpha)\sin^{-1}\frac{u-\alpha}{\beta}\Big|_{u_1}^{u_2}$$

$$= (1+3/2\,\alpha)\,[\sin^{-1}1-\sin^{-1}(-1)] = \pi+\frac{3}{2}\pi\alpha$$

This the angle between sucessive perihelion / 2

$\Rightarrow$ the procession angle per-period.

$$\Delta\phi = 2\delta\phi-2\pi = 3\pi\epsilon\cdot\alpha = 3\pi\cdot\frac{2G_NM}{c^2}\left(\frac{1}{r_1}+\frac{1}{r_2}\right)\frac{1}{2}$$

$$= 3\pi\cdot\frac{G_NM}{c^2}\cdot\left(\frac{1}{r_1}+\frac{1}{r_2}\right) = \frac{3\pi}{2}\,r_s\left(\frac{1}{r_1}+\frac{1}{r_2}\right)$$

r_s (sun) $\simeq 3\,km$

Mercury: $a \simeq 5.8 \times 10^7\,km \quad e = 0.2$

$$\rho = \frac{p}{1+e\cos\theta} \qquad r_1 = \frac{p}{1-e} \quad \Rightarrow 2a = p\,\frac{2}{1-e^2}$$

$$r_2 = \frac{p}{1+e} \qquad a = \frac{p}{1-e^2}$$

$$\frac{1}{r_1} + \frac{1}{r_2} = \frac{2}{p} = \frac{2}{a(1-e^2)} \qquad \Rightarrow \frac{1}{p} = \frac{1}{a(1-e^2)}$$

$$\Rightarrow \Delta\phi = 3\pi \frac{r_s}{a} \frac{1}{1-e^2} = 3\pi \cdot \frac{3}{5.8 \times 10^7} \frac{1}{1-0.04}$$

$$\simeq 4.8 \times 10^{-7}$$

Mercury ~ 88 days. 1 centry $\simeq$ 415 round

$$\Rightarrow \Delta\phi/100\text{ years} \simeq \frac{4.8 \times 10^{-7} \times 415}{0.96} \approx 2.08 \times 10^{-4}$$

1 rad $\simeq 57.3° \simeq 2.06 \times 10^5$ 秒

$$\Rightarrow \boxed{\Delta\phi/100\text{ years} \simeq 43''}$$