Lect 12

Cosmological models and big bang

§1 Friedman - Robertson - Walker line-element

$$d\tau^2 = dt^2 - R^2(t)\left(\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\, d^2\phi)\right)$$

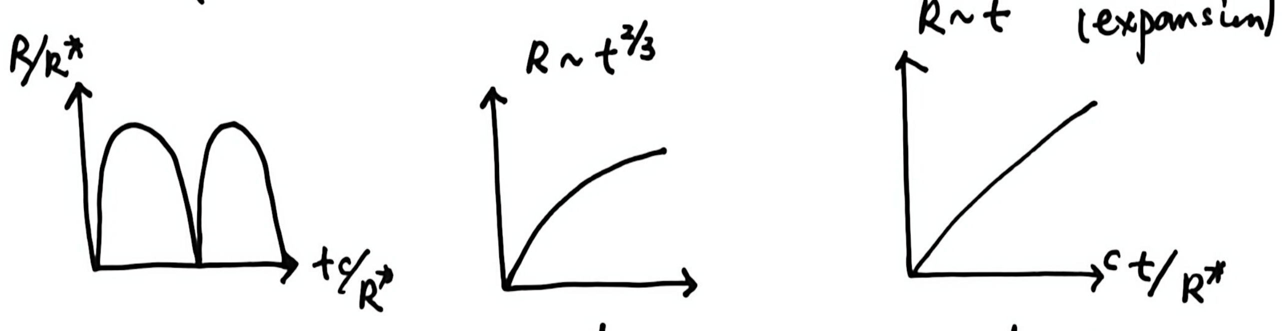
§2. Hubble's law

H - debate
- 73.5 km/s/Mpc
- 67.4 km/s/Mpc

$$\frac{\lambda(\Delta\chi)-\lambda(0)}{\lambda(0)} \simeq \frac{1}{R}\frac{dR}{dt}\cdot\Delta\chi = H\Delta\chi$$

§. Universe expansion ($k = +1, 0, -1$)

$$\frac{1}{R^2}\left(\frac{dR}{c\,dt}\right)^2 + \frac{k}{R^2} = \frac{8\pi}{3c^4}G_N T_{00}$$



$k=1$ (oscillation) $\quad$ $k=0$ $\quad$ $k=-1$

§ Big bang theory - Gamow

CMB - temp: Thermodynamics

$\to T_{CMB} \simeq 7.3\,K.$ — crude but very cute!

§ Homogeneity and isotropy

Our universe is homogeneous and isotropic over a length scale of 10^7 parsecs. (1 parsec = 3.2 ly). We write down

$$d\tau^2 = dt^2 - ds^2 \leftarrow ds^2 \text{ must exhibit homogeneity and isotropy}$$

$$R_{\mu\nu\lambda\rho} = K(g_{\mu\lambda} g_{\nu\rho} - g_{\mu\rho} g_{\nu\lambda})$$

A locally flat coordinate system at a point will be Minkowski only, Minkowski in invariant under rotation.

g-tensor structure in consistent with the symmetry of R

K is a constant dual to homogeneity.

We choose $R^{(3)}_{\mu\nu\lambda\rho} = K(g^{(3)}_{\mu\lambda} g^{(3)}_{\nu\rho} - g^{(3)}_{\mu\rho} g^{(3)}_{\nu\lambda})$

and $d\tau^{2\,(3)} = A(r)\,dr^2 + r^2 d\Omega$

where $A(r) = e^{\lambda(r)}$

- Example (3D space of constant curvature)

$$R_{\mu\nu\lambda\rho} = k\,(g_{\mu\lambda}\, g_{\nu\rho} - g_{\mu\rho}\, g_{\nu\lambda})$$

Consider an isotropic 3D space, $dz^2 = A(r)\, dr^2 + r^2 d\Omega^2$

$$g_{11} = A(r), \quad g_{22} = r^2 \quad g_{33} = r^2 \sin^2\theta$$

According to

$$\frac{d^2 x^\nu}{dz^2} + \Gamma^\nu_{\lambda\rho} \frac{dx^\lambda}{dz}\frac{dx^\rho}{dz} = 0$$

$$g_{\mu\nu}\frac{d^2x^\nu}{dz^2} + 1/2\,(\partial_\lambda g_{\rho\mu} + \partial_\rho g_{\mu\lambda} - \partial_\mu g_{\lambda\rho})\frac{dx^\lambda}{dz}\frac{dx^\rho}{dz} = 0$$

$$\mu = 1 \Rightarrow A(r)\frac{d^2x^1}{dz^2} + 1/2\,(\partial_\lambda g_{\rho 1} + \partial_\rho g_{1\lambda} - \partial_1 g_{\lambda\rho})\frac{dx^\lambda}{dz}\frac{dx^\rho}{dz} = 0$$

$$A(r)\frac{d^2x^1}{dz^2} + \partial_1 g_{11}\left(\frac{dx^1}{dz}\right)^2 - 1/2\left[(\partial_1 g_{11})\left(\frac{dx^1}{dz}\right)^2 + \partial_1 g_{22}\left(\frac{dx^2}{dz}\right)^2 + \partial_1 g_{33}\left(\frac{dx^3}{dz}\right)^2\right] = 0$$

$$\frac{d^2x^1}{dz^2} + \frac{1}{2}\frac{A'}{A}\left(\frac{dx^1}{dz}\right)^2 - 1/2\left(\frac{2r}{A}\right)\left(\frac{dx^2}{dz}\right)^2 - 1/2\left(\frac{2r\sin^2\theta}{A}\right)\left(\frac{dx^3}{dz}\right)^2 = 0$$

$$\frac{d^2x^1}{dz^2} + \frac{1}{2}\frac{A'}{A}\left(\frac{dx^1}{dz}\right)^2 - \frac{r}{A}\left(\frac{dx^2}{dz}\right)^2 - \frac{r\sin^2\theta}{A}\left(\frac{dx^3}{dz}\right)^2 = 0$$

$$\boxed{\Gamma^1_{11} = \frac{A'}{2A}, \quad \Gamma^1_{22} = -\frac{r}{A}, \quad \Gamma^1_{33} = \frac{r}{A}\sin^2\theta}$$

③

$\mu=2$

$$r^2 \frac{d^2x^2}{d\tau^2} + \frac{1}{2}\left(\partial_\lambda g_{\rho 2} + \partial_\rho g_{2\lambda} - \underset{2}{\partial_\lambda} g_{\lambda\rho}\right) \frac{dx^\lambda}{d\tau}\frac{dx^\rho}{d\tau} = 0$$

$$r^2\frac{d^2x^2}{d\tau^2} + 2r\left(\frac{dx^2}{d\tau}\right)\left(\frac{dx^1}{d\tau}\right) - \frac{1}{2}\frac{d}{d\theta}\left(r^2\sin^2\theta\right)\frac{dx^3}{d\tau}\frac{dx^3}{d\tau} = 0$$

$$\frac{d^2x^2}{d\tau^2} + \frac{2}{r}\left(\frac{dx^1}{d\tau}\right)\left(\frac{dx^2}{d\tau}\right) - \sin\theta\cos\theta\left(\frac{dx^3}{d\tau}\right)^2 = 0$$

$$\boxed{\Gamma^2_{12} = \Gamma^2_{21} = 1/r, \qquad \Gamma^2_{33} = -\sin\theta\cos\theta}$$

$\mu=3$

$$r^2\sin^2\theta\frac{d^2x^3}{d\tau^2} + \frac{1}{2}\left(\partial_\lambda g_{\rho 3} + \partial_\rho g_{3\lambda} - \partial_3 g_{\lambda\rho}\right)\frac{dx^\lambda}{d\tau}\frac{dx^\rho}{d\tau} = 0$$

$$r^2\sin^2\theta\frac{d^2x^3}{d\tau^2} + \left(2r\sin^2\theta\,\frac{dx^1}{d\tau}\frac{dx^3}{d\tau} + 2r^2\sin\theta\cos\theta\left(\frac{dx^3}{d\tau}\right)\left(\frac{dx^2}{d\tau}\right)\right) = 0$$

$$\cancel{-\frac{1}{2}r^2\,2\sin\theta\cos\theta\left(\frac{dx^3}{d\tau}\right)^2 = 0}$$

$$\frac{d^2x^3}{d\tau^2} + \frac{2}{r}\frac{dx^1}{d\tau}\frac{dx^3}{d\tau} + 2\cot\theta\left(\frac{dx^3}{d\tau}\right)^2\left(\frac{dx^2}{d\tau}\right) = 0$$

$$\boxed{\Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r}, \qquad \Gamma^3_{23} = \Gamma^3_{32} = \cot\theta}$$

$$R_{\mu\nu\lambda\rho} = g_{\mu\sigma}\left(\partial_\lambda\Gamma^\sigma_{\nu\rho} - \partial_\rho\Gamma^\sigma_{\nu\lambda} + \Gamma^\beta_{\nu\rho}\Gamma^\sigma_{\beta\lambda} - \Gamma^\beta_{\nu\lambda}\Gamma^\sigma_{\beta\rho}\right)$$

$\mu=\lambda=1,\ \nu=\rho=2$

$$R_{1212} = g_{1\sigma}\left(\partial_1\Gamma^\sigma_{22} - \partial_2\Gamma^\sigma_{21} + \Gamma^\beta_{22}\Gamma^\sigma_{\beta 1} - \Gamma^\beta_{21}\Gamma^\sigma_{\beta 2}\right)$$

$$= A\left[\partial_r\left(\frac{-r}{A}\right) + \Gamma^1_{22}\Gamma^1_{11} - \Gamma^2_{21}\Gamma^1_{22}\right] = -A\partial_r\left(\frac{r}{A}\right) - A\frac{r}{A}\frac{A'}{2A} - A\frac{1}{r}\left(\frac{-r}{A}\right)$$

$$R_{1212} = -1 + A\frac{rA'}{A^2} - \frac{rA'}{2A} + 1 = \frac{r}{2A}A'$$

$$R_{1212} = K(g_{11}g_{22} - g_{12}g_{21}) = KAr^2$$

$$\Rightarrow \quad \frac{rA'}{2A} = KAr^2 \quad \Rightarrow \quad \frac{A'}{A^2} = 2Kr \quad \Rightarrow \left(\frac{1}{A}\right)' = -2Kr$$

$$\frac{1}{A} = \underset{\text{const}}{} - Kr^2 \quad \Rightarrow \quad A = \frac{1}{\text{const} - Kr^2}$$

when $K \to 0$, we should recover the ordinary spherical coordinate $A = 1 \Rightarrow$ const $= 1$.

$$A = \frac{1}{1-Kr^2} \quad \Rightarrow \quad \boxed{d\tau^2 = \frac{dr^2}{1-Kr^2} + r^2 d\Omega^2}$$

If $K > 0$, and $Kr^2 < 1$. we define

$$\frac{(dr)^2}{1-Kr^2} = (d\rho)^2 \quad \Rightarrow \quad \frac{1}{\sqrt{K}}\frac{dr'}{\sqrt{1-r'^2}} = d\rho$$

$$\Rightarrow \quad \sin^{-1} r' = \sqrt{K}\, d\rho \quad \Rightarrow \quad \sqrt{K}\, r = \sin\sqrt{K}\rho$$

$$r = \frac{\sin\sqrt{K}\rho}{\sqrt{K}} \quad \Rightarrow \quad d\tau^2 = d\rho^2 + \frac{1}{K}\sin^2\sqrt{K}\rho\, d\Omega^2$$

If $K < 0$, $\quad K = -|K| \quad \Rightarrow \quad r = \sinh\sqrt{|K|}\rho \big/ \sqrt{|K|} \quad d\rho^2 = \frac{dr^2}{1+|K|r^2}$

$$\boxed{d\tau^2 = d\rho^2 + \frac{1}{|K|}\sinh^2(\sqrt{|K|}\rho)\, d\Omega^2}$$

$$ds^2 = \frac{dr^2}{1-Kr^2} + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2)$$

Define $r^* = \sqrt{|K|}\, r$, and $k = K/|K| = \pm 1, 0$

$$\Rightarrow ds^2 = \frac{1}{|K|} \left(\frac{dr^{*2}}{1-k r^{*2}} + r^{*2} (d\theta^2 + \sin^2\theta \, d\phi^2) \right)$$

$\rightarrow$ ignore $*$

$$d\tau^2 = dt^2 - ds^2$$

$$= dt^2 - R^2(t) \left(\frac{dr^2}{1-kr^2} + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2) \right)$$

Friedman – Robertson – Walker line element

$R(t)$ scale factor.

§. Close universe $k = 1$

$$r = \sin\chi, \quad dr = \cos\chi \, d\chi$$

$$ds^2 = R^2(t) \left(d\chi^2 + \sin^2\chi \, (d\theta^2 + \sin^2\theta \, d\phi^2) \right) \leftarrow S^3$$

$$\left.\begin{aligned} x_1 &= R\cos\chi \\ x_2 &= R\sin\chi \sin\theta \cos\phi \\ x_3 &= R\sin\chi \sin\theta \sin\phi \\ x_4 &= R\sin\chi \cos\theta \end{aligned}\right\} \Rightarrow \sum_{i=1}^{4} x_i^2 = R^2$$

$$dx_1^2 + \cdots dx_4^2 = ds^2$$

· Flat universe $k=0$: $\quad Rr \rightarrow$

$$x_1 = Rr\sin\theta\cos\phi$$
$$x_2 = Rr\sin\theta\sin\phi$$
$$x_3 = Rr\cos\phi$$

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2$$

§ open universe, $k=-1$

$r = \sinh\chi$, $\quad dr = \cosh\chi\, d\chi$

$$\Rightarrow \quad ds^2 = R^2\left[\frac{\cosh^2\chi\, d\chi^2}{\cosh^2\chi} + \sinh^2\chi\,(d\theta^2 + \sin^2\theta\, d\phi^2)\right]$$

$$= R^2\left(d\chi^2 + \sinh^2\chi\,(d\theta^2 + \sin^2\theta\, d\phi^2)\right)$$

$$\left.\begin{aligned} x_1 &= R\cosh\chi \\ x_2 &= R\sinh\chi\sin\theta\cos\phi \\ x_3 &= R\sinh\chi\sin\theta\sin\phi \\ x_4 &= R\sinh\chi\cos\theta \end{aligned}\right\} \Rightarrow \quad x_1^2 - x_2^2 - x_3^2 - x_4^2 = R^2(t)$$

and $ds^2 = -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^3$

3d surface of a pseudosphere : open universe.

Hubble's law:

$$k = \pm 1 \quad \Rightarrow \quad d\tau^2 = dt^2 - R^2(t)\left(d\chi^2 + \begin{matrix}\sin^2\chi \\ \sinh^2\chi\end{matrix}\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)\right)$$

where χ, θ, ϕ: co-moving ~~frame~~ coordinates

As the galaxies expand, ~~the distance b~~ the metric changes, but χ, θ, ϕ do not.

Consider the radial 3d distance between 2 galaxies

$$l(t) = \int dl = \int d\chi\, R(t) = R(t)\,\Delta\chi$$

$$\frac{dl(t)}{dt} = \frac{dR(t)}{dt}\Delta\chi = \frac{1}{R(t)}\frac{dR(t)}{dt}R(t)\Delta\chi = \frac{\dot{R}}{R}\cdot l(t)$$

$$\Rightarrow v(t) = \frac{dl(t)}{dt} = H\ l(t) \quad \Rightarrow \boxed{H = \dot{R}/R \quad \text{Hubble's constant}}$$

$$d\tau^2 = dt^2 - R^2(t)\,d\chi^2 = R^2(t)\left(d\eta^2 - d\chi^2\right)$$

we define $dt = R\,d\eta$: The null geodesics.

$$d\eta^2 - d\chi^2 = 0$$

$$\Delta\chi(\eta) = \eta - \eta_0$$

If two signals are emitted from the same point at time η_0 and $\eta_0 + \Delta\eta_0$, then these two world lines would be

$$\Delta\chi = \eta - \eta_0$$

$$\Delta\chi = \eta - \eta_0 - \Delta\eta_0$$

Two signals arrive at a given coordinate χ with a time difference $\Delta\eta(\chi) = \Delta\eta_0$ = constant independent of coordinate

The proper time interval

$$\Delta\tau = R\Delta\eta \quad \Rightarrow \quad \frac{R(\eta)}{\Delta\tau} = \frac{1}{\Delta\eta} = \text{const}$$

or $R(\eta)\,\nu(\Delta\chi) = \text{const}$

In an expanding universe R increase. The ν must decrease

$$R(\eta)\,\nu(\Delta\chi) = R(\eta_0)\,\nu(\Delta\chi = 0)$$

$$\nu(\Delta\chi) = \nu(0).\,R(\eta_0)/R(\eta) = \nu(0)\,\frac{R(\eta - \Delta\chi)}{R(\eta)}$$

if $\chi \ll 1$,

$$\nu(\Delta\chi) \simeq \nu(0)\,\frac{1}{R(\eta)}\left(R(\eta) - \Delta\chi\,\frac{dR(\eta)}{d\eta}\right)$$

$$\frac{\nu(\Delta\chi) - \nu(0)}{\nu(0)} \simeq -\frac{1}{R(\eta)}\,\frac{dR(\eta)}{d\eta}\,\Delta\chi$$

$$dt = R\,d\eta \quad \Rightarrow \quad \frac{\nu(\Delta\chi) - \nu(0)}{\nu(0)} = -\frac{d\eta}{dt}\,\frac{dR}{d\eta}\,\Delta\chi = -\dot{R}\Delta\chi$$

wavelength
$$\boxed{\frac{\lambda(\Delta\chi) - \lambda(0)}{\lambda(0)} \simeq \dot{R}\Delta\chi}$$

$$\Rightarrow \quad \frac{\lambda(\circ\chi)-\lambda(0)}{\lambda(0)} = \dot{R}\circ\chi = \frac{\dot{R}}{R}\cdot R\circ\chi = H\,l(t) = v$$

$\Rightarrow$ Another form of Hubble's law - red shift

The more recent universe (~~n~~ galaxy nearby)

$H \simeq 73.5 \pm 0.81$ km/s/Mpc

Early universe $H \simeq 67.2$ km/s/Mpc (theory)

{ measurment $H \approx 67.9 \pm 4$

gravity-wave, microwave background.

Debate on $H \simeq 73.5$ or 67.4

$T = H^{-1}$ ~ approximate age of the universe

if $H = 70$ km/s/Mpc, $\to T \simeq 1.4$ billion years

Hubble distance $c/H_0 \simeq 1.4$ billion light years.

visible universe

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Evolution equation

$$d\tau^2 = dt^2 - R^2(t)\left(\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\right)$$

$$\boxed{L = \frac{1}{2}\dot{t}^2 - \frac{1}{2}\frac{R^2}{1-kr^2}\dot{r}^2 - \frac{1}{2}R^2 r^2\dot{\theta}^2 - \frac{1}{2}R^2 r^2 \sin^2\theta\,\dot{\phi}^2}$$

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{t}}\right) = \frac{\partial L}{\partial t} \Rightarrow \ddot{t} + R\frac{dR}{dt}\left(\frac{1}{1-kr^2}\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2\right) = 0$$

Compare $\cancel{g_{\mu\nu}\frac{d^2x^\nu}{d\tau^2} + \frac{1}{2}(\partial_\lambda g_{\rho\mu}} + \partial_\rho g_{\mu\lambda} \cancel{- \partial_\mu g_{\lambda\rho}})\frac{dx^\lambda}{d\tau}\frac{dx^\rho}{d\tau} = 0$

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} = 0$$

$$\boxed{\Gamma^0_{11} = R\frac{dR}{dt}\frac{1}{1-kr^2}, \quad \Gamma^0_{22} = R\frac{dR}{dt}r^2, \quad \Gamma^0_{33} = R\frac{dR}{dt}r^2\sin^2\theta}$$

or $\Gamma^0_{ij} = \frac{1}{R}\frac{dR}{dt}g_{ij}$.

r- equation $\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{\partial L}{\partial r}$ $\qquad \frac{d}{d\tau}\left(\frac{R^2}{1-kr^2}\dot{r}\right) = \frac{\dot{r}^2 R^2 kr}{(1-kr^2)^2} + \cdots$

$$-\frac{R^2}{1-kr^2}\ddot{r} - 2R\frac{dR}{dt}\frac{1}{1-kr^2}\dot{t}\dot{r} - \frac{2kR^2r}{(1-kr^2)^2}\dot{r}^2$$

$$+\frac{kR^2r}{(1-kr^2)^2}\dot{r}^2 + R^2r\dot{\theta}^2 + R^2 r\sin^2\theta\,\dot{\phi}^2 = 0$$

$$\ddot{r} + \frac{2RdR}{Rdt}\dot{t}\dot{r} + \frac{kr}{1-kr^2}\dot{r}^2 - r(1-kr^2)(\dot{\theta}^2 - \sin^2\theta\,\dot{\phi}^2) = 0$$

$$\boxed{\begin{array}{ll} \Gamma^1_{01} = \Gamma^1_{10} = \dfrac{1}{R}\dfrac{dR}{dt}, & \Gamma^1_{11} = \dfrac{kr}{1-kr^2} \\ \Gamma^1_{22} = -r(1-kr^2) & \Gamma^1_{33} = -r(1-kr^2)\sin^2\theta \end{array}}$$

Similarly θ-equation

$$-R^2r^2\ddot{\theta} - 2Rr^2\frac{dR}{dt}\dot{t}\dot{\theta} - 2R^2r\dot{r}\dot{\theta} + R^2r^2\sin\theta\cos\theta\,\dot{\phi}^2 = 0$$

$$\ddot{\theta} + \frac{2}{R}\frac{dR}{dt}\dot{t}\dot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} - \sin\theta\cos\theta\,\dot{\phi}^2 = 0$$

$$\boxed{\Gamma^2_{02} = \Gamma^2_{20} = \frac{1}{R}\frac{dR}{dt}, \quad \Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r}, \quad \Gamma^2_{33} = -\sin\theta\cos\theta}$$

ϕ-equation

$$-R^2r^2\sin^2\theta\,\ddot{\phi} - 2Rr^2\sin^2\theta\frac{dR}{dt}\dot{t}\dot{\phi} - 2R^2\sin^2\theta\, r\dot{r}\dot{\phi}$$
$$-2R^2r^2\sin\theta\cos\theta\,\dot{\theta}\dot{\phi} = 0$$

$$\ddot{\phi} + \frac{2}{R}\frac{dR}{dt}\dot{t}\dot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} + 2\cot\theta\,\dot{\theta}\dot{\phi} = 0$$

$$\boxed{\Gamma^3_{03} = \Gamma^3_{30} = \frac{1}{R}\frac{dt}{dt}, \quad \Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r}, \quad \Gamma^3_{23} = \Gamma^3_{32} = \cot\theta}$$

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu} = \partial_\lambda \Gamma^\lambda_{\mu\nu} - \partial_\nu \Gamma^\lambda_{\mu\lambda} + \Gamma^\lambda_{\lambda\sigma}\Gamma^\sigma_{\mu\nu} - \Gamma^\lambda_{\nu\sigma}\Gamma^\sigma_{\nu\lambda}$$

$$R_{00} = \partial_\lambda \Gamma^\lambda_{00} - \partial_0 \Gamma^\lambda_{0\lambda} + \Gamma^\lambda_{\lambda\sigma}\Gamma^\sigma_{00} - \Gamma^\lambda_{0\sigma}\Gamma^\sigma_{0\lambda}$$

$$= -\partial_0(\Gamma^1_{10} + \Gamma^2_{20} + \Gamma^3_{30}) - \Gamma^1_{01}\Gamma^1_{01} - \Gamma^2_{02}\Gamma^2_{02} - \Gamma^3_{03}\Gamma^3_{03}$$

$$= -3\frac{d}{dt}\left(\frac{1}{R}\frac{dR}{dt}\right) - 3\left(\frac{1}{R}\frac{dR}{dt}\right)^2 = -\frac{3}{R}\frac{d^2R}{dt^2}$$

$$R_{11} = \partial_\lambda \Gamma^\lambda_{11} - \partial_1 \Gamma^\lambda_{1\lambda} + \Gamma^\lambda_{\lambda\sigma}\Gamma^\sigma_{11} - \Gamma^\lambda_{1\sigma}\Gamma^\sigma_{1\lambda}$$

$$= -\partial_1(\Gamma^1_{11} + \Gamma^2_{21} + \Gamma^3_{31}) + \partial_0\Gamma^0_{11} + \partial_1\Gamma^1_{11}$$

$$+ (\Gamma^1_{10} + \Gamma^2_{20} + \Gamma^3_{30})\Gamma^0_{11} + (\Gamma^1_{11} + \Gamma^2_{21} + \Gamma^3_{31})\Gamma^1_{11}$$

$$- \Gamma^0_{11}\Gamma^1_{10} - \Gamma^1_{10}\Gamma^0_{11} - \Gamma^1_{11}\Gamma^1_{11} - \Gamma^2_{12}\Gamma^2_{12} - \Gamma^3_{13}\Gamma^3_{13}$$

$$= \frac{2}{r^2} + \frac{1}{1-kr^2}\left(\frac{d}{dt}\left(R\frac{dR}{dt}\right) + \left(\frac{dR}{dt}\right)^2 + 2k\right) - \frac{2}{r^2}$$

$$= \frac{1}{1-kr^2}\left(2\left(\frac{dR}{dt}\right)^2 + R\frac{d^2R}{dt^2} + 2k\right)$$

$$R_{22} = \partial_\lambda \Gamma^\lambda_{22} - \partial_2 \Gamma^\lambda_{2\lambda} + \Gamma^\lambda_{\lambda\sigma}\Gamma^\sigma_{22} - \Gamma^\lambda_{2\sigma}\Gamma^\sigma_{2\lambda}$$

$$= \partial_1\Gamma^1_{22} - \partial_2\Gamma^3_{23} + \partial_0\Gamma^0_{22} + (\Gamma^1_{10} + \Gamma^2_{20} + \Gamma^3_{30})\Gamma^0_{22}$$

$$+ (\Gamma^1_{11} + \Gamma^2_{21} + \Gamma^3_{31})\Gamma^1_{22} - \Gamma^0_{22}\Gamma^2_{20} - \Gamma^1_{22}\Gamma^2_{21} - \Gamma^2_{20}\Gamma^0_{22}$$

$$= \operatorname{cosec}^2\theta + r^2\frac{d}{dt}\left(R\frac{dR}{dt}\right) - (1-3kr^2) + r^2\left(\frac{dR}{dt}\right)^2$$

$$+ r(1-kr^2)\left(-\frac{kr}{1-kr^2} - 1/r\right) + \frac{1}{r}r(1-kr^2) - \cot^2\theta$$

$$= r^2\left(2\left(\frac{dR}{dt}\right)^2 + R\,d^2R/dt^2 + 2k\right)$$

$$R_{33} = \partial_\lambda \Gamma^\lambda_{33} - \partial_3 \Gamma^\lambda_{\lambda 3} + \Gamma^\lambda_{\lambda\sigma}\Gamma^\sigma_{33} - \Gamma^\lambda_{3\sigma}\Gamma^\sigma_{3\lambda}$$

$$= (\partial_0 \Gamma^0_{33} + \partial_1 \Gamma^1_{33} + \partial_2 \Gamma^2_{33}) + (\Gamma^1_{10} + \Gamma^2_{20} + \Gamma^3_{30})\Gamma^0_{33}$$

$$+ (\Gamma^1_{11} + \Gamma^2_{21} + \Gamma^3_{31})\Gamma^1_{33} + \Gamma^3_{32}\Gamma^2_{33} - \Gamma^0_{33}\Gamma^3_{30} - \Gamma^1_{33}\Gamma^3_{31}$$

$$- \Gamma^2_{33}\Gamma^3_{32} - \Gamma^3_{30}\Gamma^0_{33} - \Gamma^3_{31}\Gamma^1_{33} - \Gamma^3_{32}\Gamma^2_{33}$$

$$= r^2 \sin^2\theta \frac{d}{dt}\left(R\frac{dR}{dt}\right) - \sin^2\theta\,(1-3kr^2) - (\cos^2\theta - \sin^2\theta)$$

$$+ r^2\sin^2\theta\left[\left(\frac{dR}{dt}\right)^2 - k\right] + \cot\theta \sin\theta \cos\theta$$

$$= r^2\sin^2\theta\left(2\left(\frac{dR}{dt}\right)^2 + R\frac{d^2R}{dt^2} + 2k\right)$$

$$\Rightarrow R_{00} = -\frac{3}{R}\frac{d^2R}{dt^2}, \qquad R_{ij} = \frac{1}{R^2}\left(2\left(\frac{dR}{dt}\right)^2 + R\frac{d^2R}{dt} + 2k\right)g_{ij}$$

$$R = g^{00}R_{00} + g^{ij}R_{ij}$$

$$= -\left(-\frac{3}{R}\frac{d^2R}{dt^2}\right) + \frac{3}{R^2}\left(2\left(\frac{dR}{dt}\right)^2 + R\frac{d^2R}{dt} + 2k\right)$$

$$= 6\left(\frac{1}{R^2}\left(\frac{dR}{dt}\right)^2 + \frac{1}{R}\frac{d^2R}{dt^2} + \frac{k}{R^2}\right)$$

$$G_{00} = R_{00} - \frac{1}{2}g_{00}R = 8\pi G_N T^{\mu\nu}_{00}/c^4$$

$$-\frac{3}{R}\frac{d^2R}{dt^2} - \frac{1}{2}(-)\cdot 6\left(\frac{1}{R^2}\left(\frac{dR}{dt}\right)^2 + \frac{1}{R}\frac{d^2R}{dt^2} + \frac{k}{R^2}\right) = \frac{8\pi G_N T^{\mu\nu}_{00}}{c^4}$$

restore c

$$\boxed{\frac{1}{R^2}\left(\frac{dR}{c\,dt}\right)^2 + \frac{k}{R^2} = \frac{8\pi}{3}G_N T_{00}/c^4}$$

$$T^{00} = T_{00} = \rho c^2 \qquad G_N = \frac{c^3}{\hbar} l_p^2$$

$$T^{\mu 0}_{(x)} = \sum_n p^\mu_n \, \delta^{(3)}(x - x^\mu_n(t)) \sim \rho c^2$$

$$[G \cdot T_{00} / c^4] \sim \frac{c^3}{\hbar} l_p^2 \cdot \rho c^2 / c^4 = \frac{l_p^2 \cdot p \cdot c}{p \cdot l^3 c^2 \cdot t} = \frac{1}{l \cdot ct}$$

$$\sim \frac{1}{l^2}. \quad \rightarrow \text{unit correct!}$$

$$\boxed{\frac{1}{R^2}\left(\frac{dR}{cdt}\right)^2 + \frac{k}{R^2} = \frac{8\pi G}{3c^4} T_{00} = \frac{8\pi l_p^2}{3\hbar} \rho c}$$

Case 1 : $k = 1 \qquad \rho = \frac{M}{2\pi^2 R^3} \longrightarrow$ $x_1^2 + x_2^2 + x_3^2 + x_4^2 = R^2$

$S^3 : 2\pi^2$

$$\Rightarrow \frac{1}{R^2}\left(\frac{dR}{cdt}\right)^2 + \frac{1}{R^2} = \frac{4GMc^2}{3\pi R^3 c^4}$$

$$c\,dt = R\,d\eta \qquad \frac{d\eta}{dt} = \frac{1}{R}$$

$$\frac{1}{R^2}\left(\frac{dR}{R d\eta}\right)^2 + \frac{1}{R^2} = \frac{4GM}{3\pi R^3 c^2}$$

$$\left(\frac{dR}{R d\eta}\right)^2 + 1 = \frac{4GM}{3\pi c^2 R} \leftarrow \text{define } R^* = \frac{2GM}{3\pi c^2}$$

$$\frac{dR}{d\eta} = \pm\sqrt{R(2R^* - R)}$$

$$R' = R - R^* \Rightarrow \frac{dR'}{d\eta} = \pm\sqrt{R^{*2} - R'^2}$$

$$\frac{d\left(R'/R^*\right)}{\sqrt{1-\left(\frac{R'}{R^*}\right)^2}} = \pm d\eta \Rightarrow$$

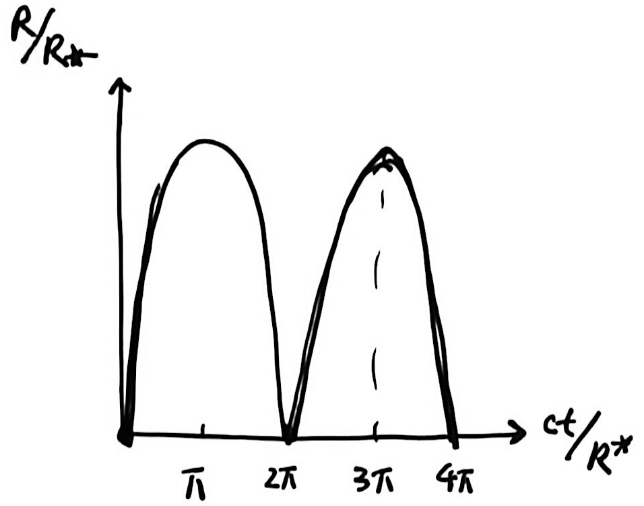
For later convenience

$$\underbrace{\frac{R'}{R^*}}_{\text{Choose}} = -\cos\eta$$

$\sin(\eta - \pi/2)$ the same.

$$\Rightarrow R(\eta) = R_*(1-\cos\eta)$$

$$c\,dt = R_*(1-\cos\eta)\,d\eta \Rightarrow ct(\eta) = R_*(\eta - \sin\eta)$$



$$\frac{dR}{c\,dt} = \frac{dR/d\eta}{c\,dt/d\eta}$$

$$= \frac{\sin\eta}{1-\cos\eta} = \cot\frac{\eta}{2}$$

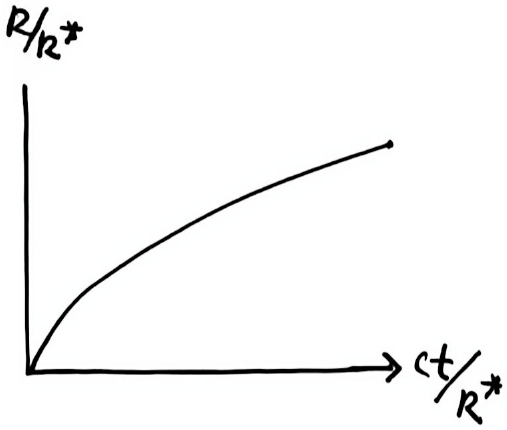
$k=1$, the universe is closed and oscilate periodically.

② $k=0$, $\frac{1}{R^2}\left(\frac{dR}{c\,dt}\right)^2 = \frac{4GM}{3\pi R^3 c^2} \Rightarrow \frac{dR}{c\,dt} = \sqrt{\frac{2R^*}{R}}$

$$\frac{dR^{\frac{3}{2}}}{c\,dt} = \left(\frac{9R^*}{2}\right)^{1/2} \Rightarrow R^{3/2}(t) = \left(\frac{9R^*}{2}\right)^{1/2}\cdot(ct)$$

$$\boxed{R(t) = (ct)^{\frac{2}{3}}\left(\frac{9R^*}{2}\right)^{1/3}}$$

$$\frac{d\eta}{dt} = \frac{1}{R} \Rightarrow \begin{array}{l} R(\eta) = \frac{R^*}{2}\eta^2 \\ ct(\eta) = \frac{R^*}{6}\eta^3 \end{array}$$



flat and open universe.

③ $k=-1$: $$\frac{1}{R^2}\left(\frac{dR}{cdt}\right)^2 - \frac{1}{R^2} = \frac{2R^*}{R^3}$$

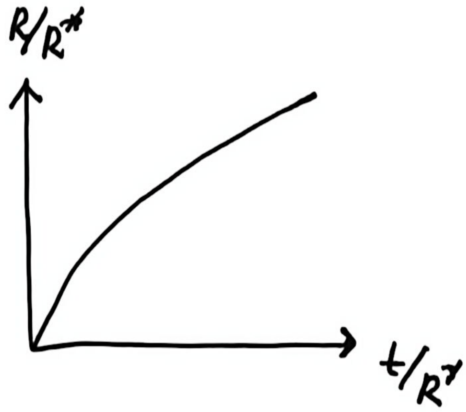
$$\left(\frac{dR}{Rd\eta}\right)^2 - 1 = \frac{2R^*}{R} \Rightarrow \frac{dR}{d\eta} = \sqrt{R(2R^*+R)}$$

$$R^{*\prime} = R+R^* \qquad \frac{dR'}{d\eta} = \sqrt{R'^2 - R^{*2}}$$

$$\boxed{R+R^* = R^*\cosh\eta}$$ ← initial condition $R=0$ at $\eta=0$

$$cdt = R\,d\eta = R^*(\cosh\eta - 1)\,d\eta$$

$$\boxed{cdt = R^*(\sinh\eta - \eta)}$$



as $t\to\infty$, $R\sim t$

$k=-1$, the open universe

Big-bang theory - Gamow

CMB - cosmic microwave background - relic temperature

$u_{rad} = \sigma T^4$, Stefan-Boltzmann constant

$$u = 2\int_0^\infty \frac{k^2 dk}{(2\pi)^3}\cdot 4\pi \frac{\varepsilon}{e^{\varepsilon/k_B T}-1}, \quad \text{where } \hbar\cdot k = \varepsilon/c$$

$$x = \varepsilon/k_B T \Rightarrow k = \frac{\varepsilon}{c\hbar} = \frac{k_B T}{\hbar c}x$$

$$= \int_0^\infty \frac{x^2 dx}{\pi^2}\frac{k_B T\cdot x}{e^x-1}\cdot\left(\frac{k_B T}{\hbar c}x\right)^3$$

$$= \int_0^\infty \frac{x^3 dx}{e^x-1}\cdot\frac{(k_B T)^4}{(\hbar c)^3}\cdot\frac{2}{(2\pi)^3} = \frac{k_B^4}{\hbar^3}\frac{T^4}{c^3}\cdot\frac{\pi^4}{15}\cdot\frac{1}{\pi^2}$$

$$= \frac{\pi^2 k_B^2}{15\hbar^3 c^3}T^4 = \sigma T^4$$

$$\sigma = \pi^2 k_B^4/15\hbar^3 c^3 = 7.56\times 10^{-16}\ \mathrm{J/(m^3\,K^4)}.$$

Hence $\frac{S}{V} = \frac{1}{V}\int\frac{dQ}{T} = \int_0^T \frac{du}{T} \sim \int T^{3-1}dT \sim T^3$

During adiabatic expansion, total entropy is fixed

Hence. $SR^3 \sim (RT)^3 = \text{constant}.$

For expanding universe, $k=-1$.

$$\frac{1}{R^2}\left(\frac{dR}{dt}\right)^2 - \frac{1}{R^2} = \frac{8\pi G_N}{3c^4}\, u$$

$$RT = \text{const} \qquad \frac{1}{R}dR + \frac{dT}{T} = 0 \Rightarrow \frac{dR}{dt} = -\frac{R}{T}\, dT/dt$$

$$\frac{1}{R^2}\frac{R^2}{T^2}\left(\frac{dT}{dt}\right)^2 - \frac{1}{R^2} = \frac{8\pi G_N}{3c^4}\cdot u$$

$$\left(\frac{dT}{cdt}\right)^2 - \underbrace{\frac{T^2}{R^2}}_{\text{const}\cdot T^4} = \frac{8\pi G_N}{3c^4}\cdot \sigma T^6$$

In the early stage of evolution, T^6 dominates over T^4.

$$dT/_c dt = -\left(\frac{8\pi G_N}{3c^4}\sigma\right)^{1/2} T^3$$

$$T^{-3} dT = -\left(\frac{8\pi G_N}{3c^4}\sigma\right)^{1/2} c\,dt \Rightarrow T^{-2} = 2\left(\frac{8\pi G_N}{3c^4}\sigma\right)^{1/2} t$$

$$T = \left(\frac{3c^4}{32\pi G_N \sigma}\right)^{1/4} (ct)^{-1/2}$$

$$u(t) = \sigma T^4 = \frac{3c^4}{32\pi G_N}(ct)^{-2}$$

$\boxed{\text{radiation energy dominates in the early univese}}$

(*) At current stage, our universe is presently dominated by matter. $\rho \sim 1/R^3$ $-$

$$\frac{1}{R^2}\left(\frac{dR}{dt}\right)^2 - \frac{1}{R^2} = \frac{8\pi G_N}{3c^4}\frac{Mc^2}{2\pi^2 R^3}$$

$$\longrightarrow 0$$

For large enough R, we drop the matter-part

$$\frac{1}{R^2}\left(\frac{dR}{dt}\right)^2 = \frac{1}{R^2} \Rightarrow \frac{dR}{c\,dt} = 1 \Rightarrow R(t) \sim ct$$

Hence $\rho_{\text{matter}} R^3 = \text{const} \Rightarrow \rho_{mat}(t)\, t^3 = \text{constant}$

The present density $\rho_{mat}(t_p) \simeq 2.4 \times 10^{-31}\ \text{gm/cm}^3$

$t_p = 1/H \simeq \ldots$ 1.4 billion years

$H = 70\ \text{km/s/Mpc} \Rightarrow t_p = \frac{3.0\times10^{8}}{70} \times$

$$1/H = \frac{1\ \text{Mpc}}{70\times10^3\ \text{m/s}} = \frac{3\times10^{22}}{7\times10^4}\ \text{s}$$

$$= 4.3\times10^{17}\ \text{s}$$

$$\Rightarrow \text{Const} = \rho_{mat}(t_p)\, t_p^3 \simeq 2.4\times10^{-31}\times4.3^3\times10^{51}\ \text{g}\cdot\text{s}^3/\text{cm}^3$$

$$= 1.9\times10^{22}\ \text{g}\cdot\text{s}^3/\text{cm}^3.$$

$$\Rightarrow \boxed{\rho_{mat}(t) \sim 1.9\times10^{22}\times t^{-3}\ \ \text{g}\cdot\text{s}^3/\text{cm}^3}$$

Gamow's idea: Early universe – radiation dominated

current universe — real matter dominate

Define t_*, such that

$$\cancel{\rho_{rad}(t} \quad u_{rad}(t_*) = \rho_{mat}(t_*)\, c^2$$

$$\left.\begin{array}{l} u_{rad}(t)/c^2 = \dfrac{3}{32\pi G_N} t^{-2} \qquad gm\cdot s^2/cm^3 \\[2ex] G_N = 6.67\times 10^{-8}\ cm^3/(gm\cdot s^2) \end{array}\right\} \Rightarrow \frac{u_{rad}(t)}{c^2} = 4.47\times 10^5 t^{-2}$$

$$\Rightarrow \rho_{mat}(t^*) = u_{rad}(t^*)/c^2$$

$$1.9\times 10^{22} t^{-3} = 4.47\times 10^5 t^{-2} \Rightarrow t = \frac{1.9\times 10^{22}}{4.47\times 10^5}\ s \approx 4.4\times 10^{16}\ s.$$

Temperature at t^*

$$T(t^*) = \left(\frac{3c^2}{32\pi G_N \sigma}\right)^{1/4} t_*^{-1/2} = \left(\frac{3\times 9\times 10^{16}}{32\times 3.14\times 6.67\times 10^{-11}\times 7.6\times 10^{-16}}\right)^{1/4} \cdot (4.4\times 10^{16})^{-1/2}\ K$$

$$= (5\times 10^{-3+16+11+16})^{1/4} \cdot 0.48\times 10^{-8}\ K$$

$$= 1.49\times 0.48\times 10^{10-8}\ K = \quad 72K.$$

In the matter dominated period, i.e $t = t_* \sim t_p$

$$R(t) \sim t$$

$$\frac{T(t_p)}{T(t_*)} = \frac{R(t_*)}{R(t_p)} = \frac{t_*}{t_p} \Rightarrow \boxed{T(t_p) = \frac{t_*}{t_p}\cdot T(t_*) = \frac{4.4\times 10^{16}}{4.3\times 10^{17}}\times 72K = 7.3K}$$

entropy dominated by the radiation

$$S_{tot} = S\cdot R^3 \propto T^3 R^3 = \text{fixed}$$