Lecture 20 Rethink Relativity

§ Space-time Lorentz transformation

Review of relativistic mechanics

§ Lorentz transformation for E&M fields

§ Establish relativistic electrodynamics

Ampere force revisit

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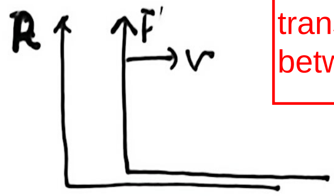
1. Space-time coordinate Lortz transform

Lortentz transformation between R and F

$$\begin{pmatrix} \Delta x' \\ c\Delta t' \end{pmatrix} = \begin{pmatrix} a & b \\ f & d \end{pmatrix} \begin{pmatrix} \Delta x \\ c\Delta t \end{pmatrix}$$



$$\frac{u}{c} = \frac{\Delta x'}{c\Delta t'} = \frac{a\Delta x + b c\Delta t}{f\Delta x + d c\Delta t} = \frac{a}{d} \frac{\frac{\Delta x}{c\Delta t} + b/a}{1 + \frac{f}{d}\frac{\Delta x}{c\Delta t}} = \frac{a}{d} \frac{\beta + b/a}{1 + \frac{f}{d}\beta}$$

compare $\frac{u}{c} = \frac{\beta_w - \beta}{1 - \beta\beta_w} \Rightarrow \frac{a}{d} = 1 \quad \frac{b}{a} = \frac{f}{d} = -\beta$

$$\Rightarrow \begin{pmatrix} \Delta x' \\ c\Delta t' \end{pmatrix} = \lambda(|v|) \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma'\beta & \gamma' \end{pmatrix} \begin{pmatrix} \Delta x \\ c\Delta t \end{pmatrix}$$

then $(\Delta x')^2 - (c\Delta t')^2 = \lambda^2(v) \left[(\Delta x)^2 - c(\Delta t)^2 \right]$

Due to the isotropy of **R**, then $\lambda(\vec{v})$ does not depend on the direction of $\pm v'$, i.e. $\lambda(|v'|)$.

Nevertheless, now the status of **R** and F are unequivalent to each other. We cannot have $\lambda^{-1}(v) = \lambda(-v)$. Let us try to obtain $\lambda = 1$ by using spatial isotropy!

We aussume the **R** frame is isotropic, hence, "c" is the upper limit of physical speed in every direction. Set a velocity $\vec{w} = (w_x, 0, w_z)$ in the **R**-frame, then in the F frame we have

$$\frac{u_z}{c} = \frac{\Delta z'/\Delta t'}{c} = \frac{\Delta z}{\lambda(\beta)(-\gamma\beta\Delta x + \gamma c\Delta t)} = \frac{\Delta z/c\Delta t}{\lambda(\beta)\,\gamma(1-\beta_x\beta)} = \frac{w_z/c}{\lambda(\beta)\gamma(1-\beta_x\beta)}$$

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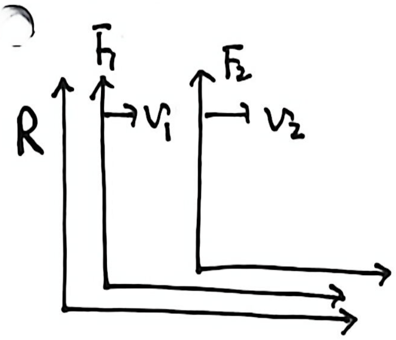
$\frac{u_x}{c} = \frac{\beta_x - \beta}{1-\beta_x \beta}$, then $\vec{u}(0,0,u_z) \longleftrightarrow \vec{w} = (v', 0, w_z)$

$$u_z = \frac{w_z}{\lambda(\beta)\,\gamma\,(1-\beta^2)} = \frac{w_z\,\gamma}{\lambda(\beta)}$$

$$w_{z,max} = \sqrt{c^2 - v'^2} = c\,\gamma'^{-1}$$

Hence, $u_{z,max} = \frac{c}{\lambda(\beta)}$.

$\boxed{\text{Due to the isotropy in the frame F, we need to set } \lambda(\beta) = 1.}$



$$\begin{pmatrix} \Delta x_1' \\ c\Delta t_1' \end{pmatrix} = \begin{pmatrix} \gamma_1 & -\gamma_1\beta_1 \\ -\gamma_1'\beta_1' & \gamma_1' \end{pmatrix} \begin{pmatrix} \Delta x \\ c\Delta t \end{pmatrix}$$

$$\begin{pmatrix} \Delta x_2' \\ c\Delta t_2' \end{pmatrix} = \begin{pmatrix} \gamma_2 & -\gamma_2\beta_2' \\ -\gamma_2'\beta_2' & \gamma_2' \end{pmatrix} \begin{pmatrix} \Delta x \\ c\Delta t \end{pmatrix}$$

Lortentz transformation between two general frames F1 and F2

$$\begin{pmatrix} \Delta x_2' \\ c\Delta t_2' \end{pmatrix} = \begin{pmatrix} \gamma_2 & -\gamma_2'\beta_2' \\ -\gamma_2'\beta_2' & \gamma_2' \end{pmatrix} \begin{pmatrix} \gamma_1' & \gamma_1'\beta_1 \\ \gamma_1'\beta_1 & \gamma_1' \end{pmatrix} \begin{pmatrix} \Delta x_1' \\ c\Delta t_1' \end{pmatrix}$$

$$= \begin{pmatrix} \gamma_{21}' & -\gamma_{21}'\beta_{21}' \\ -\gamma_{21}'\beta_{21}' & \gamma_{21}' \end{pmatrix} \begin{pmatrix} \Delta x_1' \\ c\Delta t_1' \end{pmatrix}$$

where $\beta_{21}' = \frac{\beta_2' - \beta_1'}{1-\beta_2'\beta_1'}$,

$$\gamma_{21}' = \frac{1}{\sqrt{1-\beta_{21}'^2}} = \frac{1-\beta_2'\beta_1'}{\sqrt{(1-\beta_1'^2)(1-\beta_2'^2)}}$$

$$= \gamma_1\gamma_2(1-\beta_2'\beta_1').$$

也可以通过 R 来

中介

中性粒子

- 我们采用了带电粒子在电磁场中的行为，来推导时空的洛伦兹变换。
- 时空变换是 $(\Delta x, \Delta t)$ 与 $(\Delta x', \Delta t')$ 之间的关系，独立于粒子种类以及电磁场是否存在。
- 上述洛伦兹变换同样适用于中性粒子。

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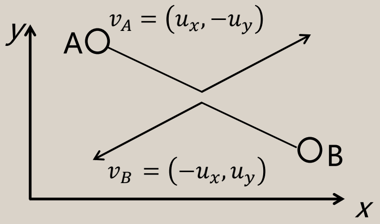
相对论力学

- **公设**：$\boldsymbol{F} = \frac{d\boldsymbol{p}}{dt}$
- **推论**：粒子的动量为 $\boldsymbol{p} = m_0\gamma_u \boldsymbol{u}$。



- **推论**：一个粒子以u的速度在惯性系F_1中运动，其受力为$\boldsymbol{F}$。在 F_2中观察时，该粒子所受的力为$\boldsymbol{F}'$。

$$F_x' = \frac{1}{1-\beta_x\beta}\left(F_x - \frac{\beta}{c}\boldsymbol{F}\cdot\boldsymbol{u}\right),\quad F_{y,z}' = \frac{F_{y,z}}{\gamma(1-\beta_x\beta)}$$

$$\beta = \frac{v}{c}, \beta_x = \frac{u_x}{c}$$

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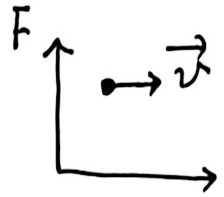
2. Review of relativistic mechanics



$\boxed{\text{F}' \text{ fixed with the particle}}$

① Proper time $\quad \Delta x' = 0, \quad c\Delta\tau = ?$

$$\begin{pmatrix} 0 \\ c\Delta\tau \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta' & \gamma' \end{pmatrix} \begin{pmatrix} \Delta x \\ c\Delta t \end{pmatrix} \Rightarrow \begin{pmatrix} \Delta x \\ c\Delta t \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta' \\ \gamma\beta' & \gamma' \end{pmatrix} \begin{pmatrix} 0 \\ c\Delta\tau \end{pmatrix} \Rightarrow \Delta t = \gamma \Delta\tau$$

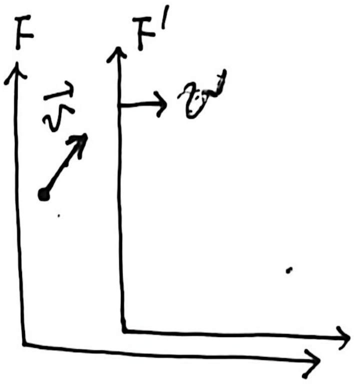
i.e. $\quad d\tau = \gamma^{-1}\, dt = \sqrt{1-\beta^2}\, dt \quad$ (by definition $d\tau$ is a scaler of Lorentz transform)

$$w^\mu = \frac{dx^\mu}{d\tau} = \left(\frac{\vec{v}}{\sqrt{1-\beta^2}}, \frac{c}{\sqrt{1-\beta^2}} \right)$$

$$p^\mu = m_0 w^\mu = \left(\frac{m_0 \vec{v}}{\sqrt{1-\beta^2}}, \frac{m_0 c}{\sqrt{1-\beta^2}} \right)$$

$$K^\mu = \frac{dp^\mu}{d\tau} = \left(\frac{dp^i}{dt}\frac{dt}{d\tau}, \frac{dp^0}{dt}\frac{dt}{d\tau} \right) = \left(\frac{F^i}{\sqrt{1-\beta^2}}, \frac{1}{c}\frac{dE}{dt}\frac{1}{\sqrt{1-\beta^2}} \right).$$

②
$$\begin{pmatrix} K'^1 \\ K'^0 \end{pmatrix} = \begin{pmatrix} \gamma' & -\gamma'\beta' \\ -\gamma'\beta' & \gamma' \end{pmatrix} \begin{pmatrix} K^1 \\ K^0 \end{pmatrix}$$



$$K'^1 = \gamma'(K^1 - \beta' K^0)$$

$$K'^0 = \gamma'(-\beta' K^1 + K^0)$$

$$\frac{F_x'}{\sqrt{1-\beta_u^2}} = \gamma' \left(\frac{F_x}{\sqrt{1-\beta^2}} - \beta' \frac{1}{c}\frac{dE}{dt}\frac{1}{\sqrt{1-\beta^2}} \right)$$

$$\frac{u_x}{c} = \frac{\beta_x - \beta'}{1-\beta_x\beta'} \qquad \frac{u_y}{c} = \frac{\beta_y}{1-\beta_x\beta'}\sqrt{1-\beta'^2} \qquad \frac{u_z}{c} = \frac{\beta_z}{1-\beta_x\beta'}\sqrt{1-\beta'^2}$$

$$1-\beta_u^2 = 1 - \frac{1}{(1-\beta_x\beta')^2}\left[(\beta_x-\beta')^2 + (\beta_y^2+\beta_z^2)(1-\beta'^2) \right] = \frac{(1-\beta^2)(1-\beta'^2)}{(1-\beta_x\beta')^2}$$

$$\frac{1}{\sqrt{1-\beta_u^2}} = \frac{1}{\sqrt{1-\beta^2}}\,\frac{1}{\sqrt{1-\beta'^2}}\,(1-\beta_x\beta')\,.$$

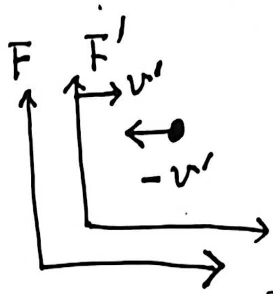
$$\Rightarrow F_x'\,(1-\beta_x\beta') = F_x - \frac{\beta'}{c}(\vec{F}\cdot\vec{v})$$

$$\boxed{F_x' = \frac{1}{1-\beta_x\beta'}\left[F_x - \frac{\beta'}{c}(\vec{F}\cdot\vec{v})\right]}$$

$$K_2' = K_2$$

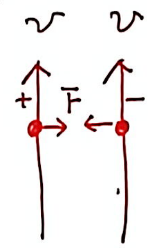
$$\frac{F_y'}{\sqrt{1-\beta_u^2}} = \frac{F_y}{\sqrt{1-\beta^2}} \Rightarrow F_y'\,\frac{1}{\sqrt{1-\beta'^2}}\,(1-\beta_x\beta') = F_y$$

$$\boxed{F_{y,z}' = \frac{\sqrt{1-\beta'^2}}{1-\beta_x\beta'}\,F_{y,z}}$$



If $\vec{v}=0$, we have simplification

$$F_x' = F_x\,,\quad F_y' = \sqrt{1-\beta'^2}\,F_y = \frac{1}{\gamma'}F_y\,,\quad F_z' = \sqrt{1-\beta'^2}\,F_z = \frac{1}{\gamma'}F_z$$



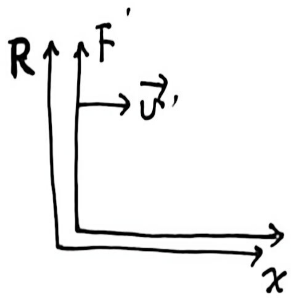
$$F = \frac{e^2}{r}\cdot\frac{1}{\gamma}$$

In a frame, the particle is moving at speed of v,

then the logitudinal force acting on it = the longitudinal force in the comoving frame.

the transforce force = $\frac{1}{\gamma}$ the transforce one in the comoving frame.

3. Lorentz transformation of E&M fields



$$E'_x = \lambda_1(|v'|)\, E_x\,, \quad B'_x = \lambda_2(|v'|)\, B_x$$

$$\begin{pmatrix} E'_y \\ B'_z \end{pmatrix} = a \begin{pmatrix} 1 & b/a \\ f/a & 1 \end{pmatrix} = a \begin{pmatrix} 1 & -\beta' \\ -\beta' & 1 \end{pmatrix} \begin{pmatrix} E_y \\ B_z \end{pmatrix}$$

$$E'^2_y - B^2_{z'} = a^2(1-\beta'^2)\,(E_y^2 - B_z^2) = \lambda^2(|v'|)\,(E_y^2 - B_z^2)$$

$$\to \begin{pmatrix} E'_y \\ B'_z \end{pmatrix} = \lambda(|v'|) \begin{pmatrix} \gamma & -\gamma\beta' \\ -\gamma\beta' & \gamma \end{pmatrix} \begin{pmatrix} E_y \\ B_z \end{pmatrix},$$

Rotation around x-axis 90°

$$E_y \to E_z\,, \quad B_z \to -B_y$$

$$\begin{pmatrix} E'_z \\ B'_y \end{pmatrix} = \lambda(|v'|) \begin{pmatrix} \gamma & \gamma\beta' \\ \gamma\beta' & \gamma \end{pmatrix} \begin{pmatrix} E_z \\ B_y \end{pmatrix}.$$

As for the E&M properties, we cannot assume relativity principle. We cannot use $\lambda^{-1}(|v'|) = \lambda(|v'|)$ to determin λ.

Consider a charge drift motion along $\hat{z}$ in the R-frame with $E_x/B_y = v_z/c$. $\leftarrow$ $\vec{F} = q\left(E_x\hat{x} + \frac{v_z}{c} B_y\, \hat{z}\times\hat{y}\right) = 0$

Then look at the same problem in the F-frame

$$\begin{pmatrix}E_z'\\ B_y'\end{pmatrix} = \lambda \begin{pmatrix}\gamma & \gamma\beta\\ \gamma\beta & \gamma\end{pmatrix}\begin{pmatrix}E_z=0\\ B_y\end{pmatrix} = \lambda\gamma B_y\begin{pmatrix}\beta\\ 1\end{pmatrix}$$

$$E_x' = \lambda_1 E_x$$

$$\Rightarrow \vec{F}' = q\left(E_x'\hat{x} + E_z'\hat{z} + \frac{1}{c}(u_x\hat{x} + u_z\hat{z})\times B_y'\hat{y}\right) = 0$$

$$\underbrace{\lambda_1 E_x}_{\frac{v_z}{c}B_y}\hat{x} + \lambda\gamma\beta B_y\hat{z} + \frac{u_x}{c}\lambda\gamma B_y\hat{z} - \frac{u_z}{c}\lambda\gamma B_y\hat{x} = 0$$

$$\Rightarrow \left(\lambda_1\frac{v_z}{c} - \frac{u_z}{c}\lambda\gamma\right)\hat{x} + \hat{z}\,\lambda\gamma\left[\beta + \frac{u_x}{c}\right]\cancel{\hat{z}} = 0$$

$$\boxed{\frac{u_z}{c} = \frac{\lambda_1}{\lambda}\frac{v_z}{c\gamma}} \qquad \boxed{\frac{u_x}{c} = -\beta}$$

According to addition of velocity:

$$\begin{cases} \dfrac{u_z}{c} = \dfrac{v_z/c}{\gamma(1-\beta\beta_x)} = \dfrac{v_z}{c\gamma} \\[2ex] \dfrac{u_x}{c} = \dfrac{\beta_x - \beta}{1-\beta_x\beta} = -\beta \end{cases} \qquad \Rightarrow \boxed{\lambda = \lambda_1}$$

Similarly, consider the drift velocity

$$\vec{F} = q\left(E_y\,\hat{y} + \frac{v_z}{c} B_x\, \hat{z}\times\hat{x}\right) = 0$$

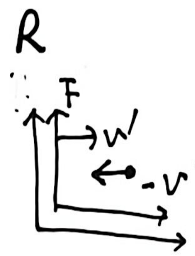
$$\frac{E_y}{B_x} = -\frac{v_z}{c},$$

then in the F-frame $B_x' = \lambda_2 B_x$

$$\begin{pmatrix} E_y' \\ B_z' \end{pmatrix} = \lambda \begin{pmatrix} \gamma' & -\gamma'\beta' \\ -\gamma'\beta' & \gamma' \end{pmatrix}\begin{pmatrix} E_y \\ 0 \end{pmatrix} = \lambda\gamma' E_y \begin{pmatrix} 1 \\ -\beta' \end{pmatrix}$$

$$B_x = -\frac{c}{v_z} E_y$$

$$\vec{F}'/q = E_y'\,\hat{y} + \left(\frac{u_x}{c}\hat{x} + \frac{u_z}{c}\hat{z}\right)\times\left(B_x'\,\hat{x} + B_z'\,\hat{z}\right)$$



$$= \left(E_y' - \frac{u_x}{c} B_z' + \frac{u_z}{c} B_x'\right)\hat{y} = 0$$

$$\Rightarrow \lambda\gamma' E_y - \frac{u_x}{c}\lambda\gamma'(-\beta')E_y + \frac{u_z}{c}\lambda_2\left(\frac{-c}{v_z}\right)E_y = 0$$

$$\lambda\gamma'\left(1 + \frac{u_x}{c}\beta'\right) = \frac{u_z}{v_z}\lambda_2$$

$$\lambda\gamma'(1-\beta'^2) = \frac{1}{\gamma'}\lambda_2$$

$$\Rightarrow \lambda = \lambda_2$$

$$\boxed{\begin{aligned} \frac{u_x}{c} &= -\beta' \\ \frac{u_z}{c} &= \frac{v_z}{c\gamma'} \end{aligned}}$$

Now we unify $\lambda = \lambda_1 = \lambda_2$. Next we use the force relation, put a static charge in F, set $\begin{cases} \vec{E} = E_x\,\hat{x} \\ \vec{B} = 0 \end{cases}$

then in F' frame, $\begin{cases} \vec{E}' = E_x'\,\hat{x} \\ \vec{B}' = 0 \end{cases}$ $\Rightarrow$ $F_x' = F_x$

$$F_x = \frac{F_x'}{1-\beta'^2}(1-\beta'^2) = F_x'$$

$$\Rightarrow E_x = E_x' \quad \Rightarrow \quad \lambda(|v|) = 1$$

$$\boxed{\begin{aligned} &E_x' = E_x, && \begin{pmatrix} E_y' \\ B_z' \end{pmatrix} = \begin{pmatrix} \gamma' & -\gamma'\beta' \\ -\gamma'\beta' & \gamma' \end{pmatrix}\begin{pmatrix} E_y \\ B_z \end{pmatrix} \\ &B_x' = B_x, && \begin{pmatrix} E_z' \\ B_y' \end{pmatrix} = \begin{pmatrix} \gamma' & \gamma'\beta' \\ \gamma'\beta & \gamma' \end{pmatrix}\begin{pmatrix} E_z \\ B_y \end{pmatrix} \end{aligned}}$$

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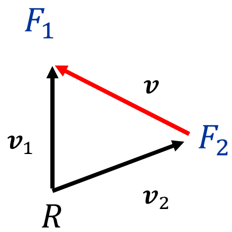
Lorentz transformation of EM fields between two general frames

Theorem 5 The Lorentz transformations of the EM fields between $\mathcal{F}_2$ and $\mathcal{F}_1$ can be formulated as follows:

$$\begin{aligned}
\mathbf{E}'_{\parallel,2} &= \mathbf{E}_{\parallel,1}, \quad \mathbf{B}'_{\parallel,2} = \mathbf{B}_{\parallel,1}, \\
\mathbf{E}'_{\perp,2} &= \gamma(\mathbf{E}_{\perp,1} + \frac{\mathbf{v}}{c} \times \mathbf{B}_{\perp,1}), \\
\mathbf{B}'_{\perp,2} &= \gamma(\mathbf{B}_{\perp,1} - \frac{\mathbf{v}}{c} \times \mathbf{E}_{\perp,1}),
\end{aligned} \qquad (47)$$

where frame $\mathcal{F}_2$ moves at $\mathbf{v} = v\hat{\mathbf{x}}$ relative to $\mathcal{F}_1$; $\parallel$ and $\perp$ represent directions parallel and perpendicular to $\mathbf{v}$, respectively.



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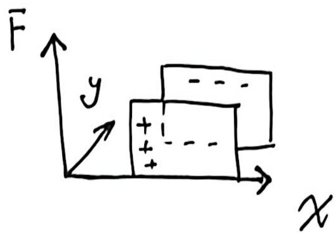
4 Gauss's law and Ampere' law revisit 10

F-frame



$$\partial_y E_y = 4\pi \rho(y)$$

$$\partial_y B_z = \frac{4\pi}{c} j_x(y)$$

$$\rightarrow \quad \rho = \frac{\Delta Q}{\Delta x \Delta y \Delta z\, c\Delta t} \cdot c\Delta t$$

$$\frac{j_x}{c} = \frac{\Delta Q}{\Delta x \Delta y \Delta z\, c\Delta t}\, \Delta x$$

$$\boxed{\begin{aligned}&\Delta V_4 \\ &= \Delta x \Delta y \Delta z\, c\Delta t\end{aligned}}$$

$$\Delta V_4' = \frac{\partial(\Delta x' \Delta y' \Delta z' t)}{\partial(\Delta x \Delta y \Delta z \Delta t)}$$

$$\Delta V_4' = \Delta V_4$$

$$\begin{pmatrix} j_x'/c \\ \rho' \end{pmatrix} = \frac{\Delta Q}{\Delta V_4'} \begin{pmatrix} \Delta x' \\ c\Delta t' \end{pmatrix} = \frac{\Delta Q}{\Delta V_4} \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} \Delta x \\ c\Delta t \end{pmatrix}$$

$$= \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} j_x/c \\ \rho \end{pmatrix}$$

$$\begin{pmatrix} \partial_y B_z'(y) \\ \partial_y E_y'(y) \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \underbrace{\partial_y \begin{pmatrix} B_z \\ E_y \end{pmatrix}}_{4\pi \begin{pmatrix} j_x/c \\ \rho \end{pmatrix}} = 4\pi \begin{pmatrix} j_x'/c \\ \rho' \end{pmatrix}.$$

more generally

$$\partial_x E_x + \partial_y E_y + \partial_z E_z = 4\pi \rho(x, y, z)$$

$$\partial_y B_z - \partial_z B_y = \frac{4\pi}{c} j_x(x, y, z)$$

$$\partial_z B_x - \partial_x B_z = \frac{4\pi}{c} j_y(x, y, z)$$

$$\partial_x B_y - \partial_y B_x = \frac{4\pi}{c} j_z(x, y, z)$$

$$\begin{pmatrix} x' \\ y' \\ z' \\ ct' \end{pmatrix} = \begin{pmatrix} \gamma & & & -\gamma\beta \\ & 1 & & \\ & & 1 & \\ -\gamma\beta & & & \gamma \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}$$

$$\begin{pmatrix} E_y' \\ B_z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} E_y \\ B_z \end{pmatrix}$$

$$\begin{pmatrix} E_z' \\ B_y' \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} E_z \\ B_y \end{pmatrix}$$

$$E_x' = E_x$$

$$B_x' = B_x$$

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$$\partial_x = \frac{\partial}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial}{\partial(ct')}\frac{\partial(ct')}{\partial x} = \gamma\frac{\partial}{\partial x'} - \gamma\beta\frac{\partial}{\partial(ct')}$$

$$\frac{\partial}{\partial x'} = \frac{\partial}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial}{\partial(ct)}\frac{\partial(ct)}{\partial x} = \gamma\frac{\partial}{\partial x} + \gamma\beta\frac{\partial}{\partial(ct)}$$

$$\frac{\partial}{\partial ct'} = \frac{\partial}{\partial x}\frac{\partial x}{\partial ct'} + \frac{\partial}{\partial ct}\frac{\partial ct}{\partial ct'} = \beta\gamma\frac{\partial}{\partial x} + \gamma\frac{\partial}{\partial(ct)}$$

$$\partial_{x'}E'_x(x',y',z',t') + \partial_{y'}E'_y(x',y',z',t') + \partial_{z'}E'_z(x',y',z',t')$$

$$= \gamma\frac{\partial}{\partial x}E_x(x,y,z,t) + \partial_y(\gamma E_y - \gamma\beta B_z) + \partial_z(\gamma E_z + \gamma\beta B_y)$$

$$= \gamma(\partial_x E_x + \partial_y E_y + \partial_z E_z) - \gamma\beta(\partial_y B_z - \partial_z B_y)$$

$$= 4\pi\left(\gamma\cdot\rho - \gamma\beta\frac{1}{c}j_x\right) = 4\pi\rho'$$ — Gauss's law

$$\partial_{y'}B'_z - \partial_{z'}B'_y = \partial_y(-\gamma\beta E_y + \gamma B_z) - \partial_z(\gamma\beta E_z + \gamma B_y)$$

$$-\frac{\partial}{\partial ct'}E'_x = -\gamma\beta(\partial_y E_y + \partial_z E_z) + \gamma(\partial_y B_z - \partial_z B_y) + \partial_x E_x$$

$$= \left(-\gamma\beta\rho + \gamma\frac{j_x}{c}\right)\cdot 4\pi = 4\pi\frac{j'_x}{c}$$ … Ampere's law

$$\partial_{x'}B'_x + \partial_{y'}B'_y + \partial_{z'}B'_z = \gamma\frac{\partial}{\partial x}B_x + \partial_y(\gamma\beta E_z + \gamma B_y) + \partial_z(-\gamma\beta E_y + \gamma B_z)$$ — Mgnetic Gass's

$$= \gamma(\nabla\cdot B) + \gamma\beta(\partial_y E_z - \partial_z E_y) = 0$$

static electric field

$$\partial_{y'}E'_z - \partial_{z'}E'_y + \frac{\partial}{\partial ct'}B'_x = \partial_y(\gamma E_z + \gamma\beta B_y) - \partial_z(\gamma E_y - \gamma\beta B_z) + \beta\gamma\frac{\partial}{\partial x}B_x$$ ← Faraday's law

$$= \gamma(\partial_y E_z - \partial_z E_y) + \gamma\beta(\partial_y B_y + \partial_z B_z + \partial_x B_x) = 0$$

$\Rightarrow$ N

Nevertheless, they're not the Maxwell eq for the dynamic E&M fields. They're for the $E(\vec{r}-\vec{v}t)$, $\vec{B}(\vec{r}-\vec{v}t)$ type.

There're no E&M wave excitations.

Nevertheless, if we view for all possible E-M

$$\begin{cases} \nabla'\cdot E' = 4\pi\rho' \\ \nabla'\cdot B' = 0 \\ \nabla'\times E' = -\frac{1}{c}\frac{\partial}{\partial t'}B' \\ \nabla\times B' = \frac{4\pi}{c}\vec{j}' + \frac{1}{c}\frac{\partial}{\partial t}E' \end{cases}$$

we do get E-M waves and it's speed is "c"!

$$\partial_{x'}B'_{y'} - \partial_{y'}B'_{x'} - \frac{\partial}{\partial ct'}E'_z$$

$$= \gamma\frac{\partial}{\partial x}(\gamma\beta E_z + \gamma B_y) - \partial_y B_x - \beta\gamma\frac{\partial}{\partial x}(\gamma E_z + \gamma\beta B_y)$$

$$= \frac{\partial}{\partial x}[\gamma^2(1-\beta^2)B_y) - \partial_y B_x = 0$$

$$\partial_{z'}B'_{x'} - \partial_x B'_z - \frac{\partial}{c\partial t'}E'_y$$

$$= \partial_z B_x - \gamma\partial_x(-\gamma\beta \cancel{E_y} + \gamma B_z) - \beta\gamma\frac{\partial}{\partial x}(\gamma\cancel{E_y} - \gamma\beta B_z)$$

Electrodynamics

- Postulate 5: In inertial frame R defined in Postulate 3, Maxwell's Eqs are established,

$$\begin{aligned}\nabla\cdot\mathbf{E} &= 4\pi\rho,\\ \nabla\cdot\mathbf{B} &= 0,\\ \nabla\times\mathbf{E} &= -\frac{\partial}{c\partial t}\mathbf{B},\\ \nabla\times\mathbf{B} &= \frac{4\pi}{c}\mathbf{j}+\frac{\partial}{c\partial t}\mathbf{E}. \end{aligned} \tag{12}$$

All solutions to Eq. (12) represent physical EM fields in frame $\mathcal{R}$. Conversely, all physical EM fields in $\mathcal{R}$ satisfy Eq. (12).

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Light speed c

- Given Lorentz transformations, once Maxwell's Eqs are set up in R-frame, then it is valid in all frames.
- EM wave velocity is just c, and universal.

$$\nabla'\cdot\mathbf{E}' = 4\pi\rho', \tag{48}$$

$$\nabla'\cdot\mathbf{B}' = 0. \tag{49}$$

Since the steady EM fields in frame $\mathcal{R}$ can transform into time-dependent ones in frame $\mathcal{F}$, the Ampère's law is augmented by the displacement current as,

$$\nabla'\times\mathbf{B}' - \frac{\partial}{c\partial t'}\mathbf{E}' = \frac{4\pi}{c}\mathbf{j}', \tag{50}$$

In the $\mathcal{R}$-frame, the steady electric field is curl free. When transformed to the $\mathcal{F}$-frame, it is augmented to Faraday's law.

$$\nabla'\times\mathbf{E}' + \frac{\partial}{c\partial t'}\mathbf{B}' = 0. \tag{51}$$

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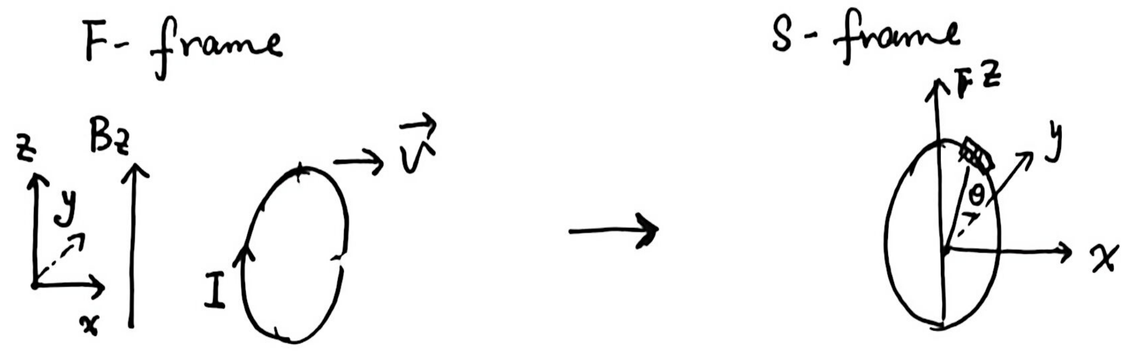
4 Ampere's force transformation:



Consider the case that only $\vec{B} \parallel \hat{z}$ in the frame, and the current ring moving along the x-direction. Now in the S- frame.

$$\begin{pmatrix} E_y^s \\ B_z^s \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} 0 \\ B_z \end{pmatrix} \Rightarrow \begin{aligned} E_y^s &= -\gamma\beta B_z \\ B_z^s &= \gamma B_z \end{aligned}$$

The line segment $dl \cdot \hat{e}_\theta$.

In the frame of S, $d\tau = \sqrt{1-\beta^2}\, dt$, hence, the current measured in frame S, $I_s = \frac{dQ}{d\tau} = \gamma \frac{dQ}{dt} = \gamma I$.

Then in the S- frame, we have

$$dF^s = \frac{1}{c} I_s \, d\vec{l} \times \vec{B}_s = \frac{1}{c}\gamma^2 I B_z \, dl \; \hat{e}_\theta \times \hat{z} = \frac{1}{c}\gamma^2 I B_z \, dl \, [\sin\theta \, \hat{y} + \cos\theta \, \hat{z}] \times \hat{z}$$

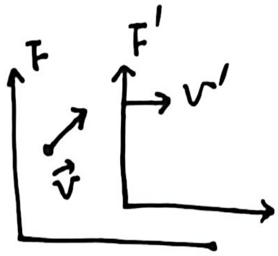
$$= \frac{1}{c}\gamma^2 B_z I \, dl \sin\theta \, \hat{x}$$

$$dP_s = \vec{E} \cdot d\vec{l} \, I_s = -\gamma^2\beta B_z \, dl \, I \sin\theta = \frac{d\varepsilon}{d\tau}$$

$$K^\mu = \frac{dp^\mu}{d\tau} = \left(\frac{F^i}{\sqrt{1-\beta^2}},\ \frac{1}{c}\frac{dE}{dt}\frac{1}{\sqrt{1-\beta^2}}\right)$$

↙

according to K's transformation



$$\frac{F_x'}{\sqrt{1-\beta_u^2}} = \gamma'\left(\frac{F_x}{\sqrt{1-\beta^2}} - \beta'\frac{1}{c}\frac{dE}{dt}\cdot\frac{1}{\sqrt{1-\beta^2}}\right)$$

↙

$\vec{v}$ in the F' frame becomes $\vec{u}$: $\frac{1}{\sqrt{1-\beta_u^2}} = \frac{1}{\sqrt{1-\beta^2}}\frac{1}{\sqrt{1-\beta'^2}}(1-\beta_x\beta')$

$$\Rightarrow F_x' = \frac{1}{1-\beta_x\beta'}\left(F_x - \frac{\beta'}{c}\frac{dE}{dt}\right)$$

Now，我们要反过来用这个公式，F系是S系，$\vec{v}=0$

in the S-frame. F'系是F系，以 $-v$ 沿 $\hat{x}$ 方向运动.

$\beta' = -v/c$

$$\Rightarrow F_x = F_x^S - \frac{\beta'}{c}\frac{dE^S}{dt}$$ ← 3个负号相乘

$$= \frac{1}{c}\left[\gamma^2 B_z I\,dl\sin\theta - \gamma^2\beta^2 B_z I\,dl\sin\theta\right]$$

$$= \gamma^2(1-\beta^2)\frac{1}{c}B_z I\,dl\sin\theta$$ ← $\gamma^2(1-\beta^2)=1$

$$= \frac{I}{c}\,d\vec{l}\times\vec{B}$$ ← Ampere's force is independent on the velocity of the current.

On the other hand,

$$\frac{1}{c}\frac{dE'}{dt'}\frac{1}{\sqrt{1-\beta_u^2}} = \gamma'\left(-\beta'\frac{F_x}{\sqrt{1-\beta^2}} + \frac{1}{c}\frac{dE}{dt}\frac{1}{\sqrt{1-\beta^2}}\right)$$

in our case. F frame is the frame S $\beta = 0$, $\beta_u = \frac{v}{c}$

F' frame is the fram F $\rightarrow$ $\beta' = -\frac{v}{c}$

$$\frac{1}{\sqrt{1-\beta_u^2}}\frac{1}{c}\frac{dE}{dt} = \frac{1}{\sqrt{1-\beta^2}}\left[-\beta' F_x^S + \frac{1}{c}\frac{dE^S}{dt}\right]$$

$$\frac{1}{c}\frac{dE}{dt} = \frac{1}{c} B_z I\, dl \sin\theta \left[\gamma^2\beta - \gamma^2\beta\right] = 0.$$

Hence, the Ampere's force does not do work.

② Next, we consider the case that $\vec{E} \parallel \hat{y}$ in the frame F.

$$\begin{pmatrix} E_y^S \\ B_z^S \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix}\begin{pmatrix} E_y \\ 0 \end{pmatrix} \Rightarrow \begin{array}{l} E_y^S = \gamma E_y \\ B_z^S = -\gamma\beta E_y \end{array}$$

then $dF_x^S = \frac{1}{c} I_s\, d\vec{l} \times \vec{B}_s = -\frac{1}{c}\gamma^2\beta E_y I\, dl \sin\theta\, \hat{x}$

$$dP^S = \vec{E}\cdot d\vec{l}\, I_s = \gamma^2 E_y I\, dl \sin\theta$$

$$\Rightarrow F_x = F_x^S + \frac{\beta}{c}\frac{dE^S}{dt} = \left(-\frac{1}{c}\gamma^2\beta + \frac{\gamma^2\beta}{c}\right) E_y I\, dl \sin\theta\, \hat{x} = 0$$

电场对电流没有力. ✓

$$\frac{1}{c}\frac{dE}{dt} = -\beta' F_x^S + \frac{1}{c}\frac{dE^S}{dt} = \left[\frac{v}{c}\left(-\frac{\beta}{c}\right) + \frac{1}{c}\right]\gamma^2 E_y I\, dl \sin\theta$$

$$\frac{1}{c}\frac{dE}{dt} = \gamma^2(1-\beta^2)\frac{E_y I\, dl}{c}\sin\theta = \frac{1}{c}\frac{dE^S}{dt}$$

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Come back to the f-frame.

$$d\vec{F} = \frac{d}{dt}\left(\frac{dm_0\,\vec{v}}{\sqrt{1-\beta^2}}\right)$$

now $\vec{v}$ is a constant
$\vec{v} = v \cdot \hat{x}$

$$= \frac{d}{dt}\left(\boxed{dm\, c^2}\right)\frac{v}{c^2}\,\hat{x}$$

$$= \vec{E}\cdot d\vec{l}\cdot I\,\frac{v}{c^2}\,\hat{x}$$

$$= \frac{E_y I v \sin\theta\, dl}{c^2}\,\hat{x}$$

If $\vec{B} = \frac{\vec{v}}{c}\times\vec{E}$, the $B_z = \frac{vE_y}{c}$

$$\Rightarrow d\vec{F} = \frac{B_z I}{c}\sin\theta\, dl\cdot\hat{x}$$

$$= \frac{I\, d\vec{l}\times\vec{B}}{c}$$

Hence, the Ampere's force just provides the $\frac{dP_x}{dt}$. Although the $\vec{v}$ does not change, m changes. Hence, force is needed!