Lecture 3 — More developement

§. Thomas precession and spin-orbit coupling

(spinor Rep for the Lorentz group)

§ Constructuct E&M and gravity via actions

§ Thomas precession

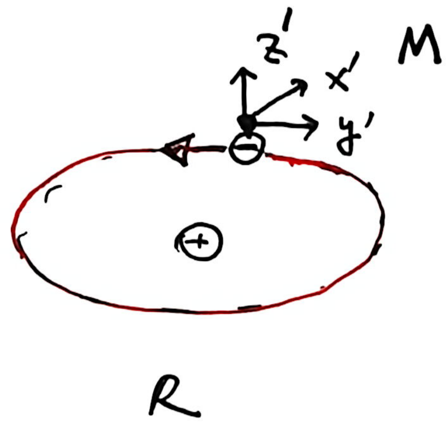
Background: Uhlenbeck & Goudsmit proposed the existance of electron spin. By assign the Lande factor in $\vec{\mu} = g\frac{e}{2mc}\vec{S}$ as $g=2$, then the anomalous Zeeman effect can be explained. But the spectrum splitting due to fine structure is too big!

Suppose we are in co-moving frame M with electron

$$\left(\frac{d\vec{S}}{dt}\right)_{\substack{\text{co-moving} \\ \text{frame} = M}} = \vec{\mu} \times \vec{B}'$$

where $\vec{B}'$ is the magnetic inductance in the co-moving frame M. Then we have $\vec{B}' \simeq \gamma\left(\vec{B} - \frac{\vec{v}}{c} \times \vec{E}\right)$

$\vec{B}$ and $\vec{E}$ are the EM field in the frame that the nucleus is at rest, denoted as R. $\vec{v}$ is electron's velocity in frame R.



Hence $\left(\frac{d\vec{S}}{dt}\right)_M = \vec{\mu} \times \left(\vec{B} - \frac{\vec{v}}{c} \times \vec{E}\right)$

This EOM can be derived via the energy of

$$\mathcal{U}' = -\vec{\mu}\cdot\left(\vec{B} - \frac{\vec{v}}{c}\times\vec{E}\right)$$

(here $e<0$ electron charge)

$$e\,\vec{E} = -\hat{r}\cdot\frac{dV}{dr} = -\frac{\vec{r}}{r}\frac{dV}{dr}$$

$$\Rightarrow \quad \mathcal{U}' = -\frac{ge}{2mc}\vec{S}\cdot\left(\vec{B} + \frac{\vec{v}}{ec}\times\vec{r}\ \frac{1}{r}\frac{dV}{dr}\right)$$

$$= -\frac{ge}{2mc}\vec{S}\cdot\vec{B} + \frac{g}{2m^2c^2}\left(\vec{S}\cdot\vec{L}\right)\left(\frac{1}{r}\frac{dV}{dr}\right)$$

$\vec{L} = m\,\vec{r}\times\vec{v}$

spin-orbit coupling

The sign of $\vec{L}\cdot\vec{S}$ term is correct. However, its magnitude is twice larger than the correct value.

Thomas pointed out if the M-frame is rotating, we will have

$$\left(\frac{d\vec{S}}{dt}\right)_F = \left(\frac{d\vec{S}}{dt}\right)_M + \vec{\omega}_T\times\ \vec{S}$$

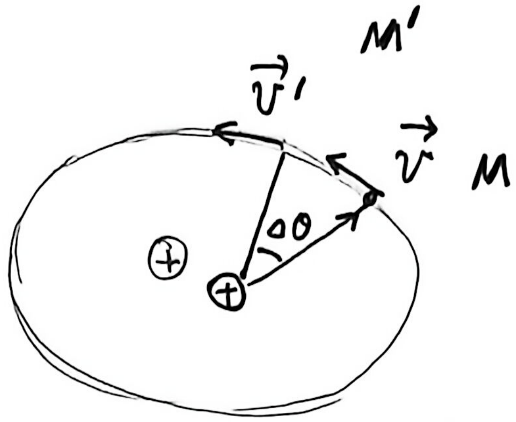
$$= \vec{S}\times\left(\frac{ge}{2mc}\vec{B}' - \vec{\omega}_T\right)$$

This corresponds to the coupling energy

$$U = \mathcal{U}' + \vec{S}\cdot\vec{\omega}_T$$

How come M is rotating?

Relative to F, the velocity of M is changing.



Here's the logic:

M: relative to F, boost along the direction of $\vec{v}$: $K(\vec{v})$

M' relative to M, is a pure boost (since force is centripetal, no-torque in M) denoted as $K(\Delta\vec{u})$

Then M' relative to F's velocity $\vec{v}'$, but

$$K(\vec{v}') \neq K(\Delta\vec{u})\, K(\vec{v}), \text{ but}$$

$$K(\vec{v}') = R(\Delta\Omega)\, K(\Delta\vec{u})\, K(\vec{v}), \text{ where } R(\Delta\Omega) \text{ is}$$

a rotation around $\hat{z}$-axis.

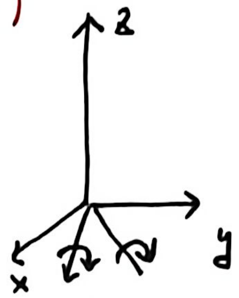
We consider $\vec{v}' \to \vec{v}$ such that $\Delta\vec{u}, \Delta\Omega$ are infinitismall. We calcute

$K(\vec{v}')\, K^{-1}(\vec{v})$ and express it as an infinitsmall boust $K(\Delta\vec{u})$ and a rotation $R(\Delta\Omega)$

$$K(\vec{v}')\, K^{-1}(\vec{v}) \simeq R(\Delta\Omega)\, K(\Delta\vec{u})$$

and $\omega_T = \Delta\Omega / \Delta t$

☆ Warm-up: Euclidean rotation (spinor Rep)



- Consider a rotation in the plane of $\hat{r}\times\hat{z}$, where $\hat{r}$ lies in the xy-plane. Rotation angle is r, define $\vec{r} = r\cdot\hat{r}$.

$$\mathcal{U}_{\hat{r}'\times z}(r') = \cos\frac{r'}{2} - i\sin\frac{r'}{2}\,\vec{\sigma}\cdot(\hat{r}'\times\hat{z})$$

$$\mathcal{U}_{\hat{r}\times z}(r) = \cos\frac{r}{2} - i\sin\frac{r}{2}\,\vec{\sigma}\cdot(\hat{r}\times\hat{z})$$

- For simplicity, we set $r = r'$, and $\Delta\vec{r} = \vec{r}' - \vec{r}$.

Then

$$\mathcal{U}_{\hat{r}'\times\hat{z}}(r)\,\mathcal{U}^{-1}_{\hat{r}\times\hat{z}}(r) = \cos\frac{r}{2}\cos\frac{r}{2} + \sin\frac{r}{2}\sin\frac{r}{2}\;\vec{\sigma}\cdot(\hat{r}'\times\hat{z})\,\vec{\sigma}\cdot(\hat{r}\times\hat{z})$$

$$+ i\cos\frac{r}{2}\sin\frac{r}{2}\,\vec{\sigma}\cdot(\hat{r}\times\hat{z}) - i\cos\frac{r}{2}\sin\frac{r}{2}\,\vec{\sigma}\cdot(\hat{r}'\times\hat{z})$$

$$[\vec{\sigma}\cdot(\hat{r}'\times\hat{z})][\vec{\sigma}\cdot(\hat{r}\times\hat{z})] = (\hat{r}'\times\hat{z})(\hat{r}\times\hat{z}) + i\,\vec{\sigma}\cdot[(\hat{r}'\times\hat{z})\times(\hat{r}\times\hat{z})]$$

- $(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = (\vec{a}\cdot\vec{c})(\vec{b}\cdot\hat{d}) - (\vec{a}\cdot\vec{d})(\vec{b}\cdot\vec{c})$

$$(\vec{a}\times\vec{b})\times(\vec{a}\times\vec{c}) = \vec{a}\,[\vec{a}\cdot(\vec{b}\times\vec{c})]$$

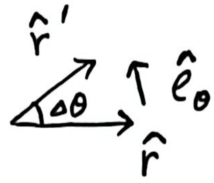
$$\Rightarrow [\vec{\sigma}\cdot(\hat{r}'\times\hat{z})][\vec{\sigma}\cdot(\hat{r}\times\hat{z})] = \hat{r}'\cdot\hat{r} + i\,\vec{\sigma}\cdot(\hat{r}'\times\hat{r})$$

$$\Rightarrow \mathcal{U}_{\hat{r}'\times\hat{z}}(r)\,\mathcal{U}^{-1}_{\hat{r}\times\hat{z}}(r) \simeq \cos^2\frac{r}{2} + \sin^2\frac{r}{2}\,[1 + i\,\sigma_z(\hat{r}'\times\hat{r})]$$

$$+ i\cos\frac{r}{2}\sin\frac{r}{2}\,\vec{\sigma}[(\hat{r}-\hat{r}')\times\hat{z}]$$

- $$= 1 - i\sin^2\frac{r}{2}\,\frac{1}{r^2}(\vec{r}\times\vec{r}')_z\,\sigma_z + i\,\frac{\sin r}{2}[(\hat{r}'-\hat{r})\times\hat{z}]\cdot\vec{\sigma}$$

set $\hat{r}' - \hat{r} = \Delta\theta \cdot \hat{e}_\theta$, $\hat{e}_\theta \times \hat{z} = \hat{r}$

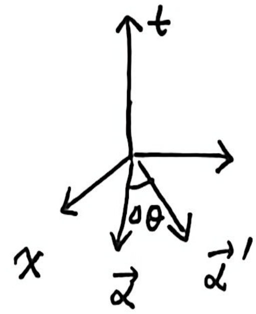


$$\frac{1}{r^2}(\vec{r} \times \vec{r}\,') = \Delta\theta \cdot \hat{z}$$

$$U_{\hat{r}'\times\hat{z}}(r)\, U^{-1}_{\hat{r}\times\hat{z}}(r) \simeq 1 - i \sin^2\frac{r}{2} \cdot \Delta\theta\, \sigma_z + i\frac{\sin r\, \Delta\theta}{2}(\vec{\sigma}\cdot\hat{r}$$

$$\simeq U_{\vec{\sigma}\cdot\hat{r}}(-\sin r\, \Delta\theta)\; U_{\hat{z}}\left[(1-\cos r)\cdot\Delta\theta\right]$$

(*) Next we will apply the similar method to Lorentz boost



Fundamental spinor

$$U(\vec{\theta}, \vec{\alpha}) = e^{-(i\vec{\theta} + \vec{\alpha})\cdot\frac{\vec{\sigma}}{2}}$$

Boost $K(\vec{\alpha}) = e^{-\vec{\alpha}\cdot\frac{\vec{\sigma}}{2}} = \cosh\frac{\alpha}{2} - \sinh\frac{\alpha}{2}(\hat{\alpha}\cdot\vec{\sigma})$

For convenience, we switch to rapidity

→ Vector Rep for $\alpha = \alpha\hat{z}$

$$\begin{pmatrix} \cosh\alpha & 0 & 0 & -\sinh\alpha \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh\alpha & 0 & 0 & \cosh\alpha \end{pmatrix} \rightarrow \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix}$$

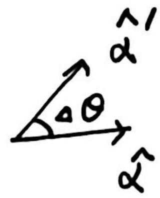
$$\Rightarrow \boxed{\tanh\alpha = \beta}$$

$$K(\vec{\alpha}')\,K^{-1}(\vec{\alpha}) = \left[\cosh\frac{\alpha}{2} - \sinh\frac{\alpha}{2}\,(\hat{\alpha}'\cdot\vec{\sigma})\right]\left[\cosh\frac{\alpha}{2} + \sinh\frac{\alpha}{2}\,(\hat{\alpha}\cdot\vec{\sigma})\right]$$

$$= \cosh^2\frac{\alpha}{2} - \sinh^2\frac{\alpha}{2}\,(\hat{\alpha}'\cdot\vec{\sigma})(\hat{\alpha}\cdot\vec{\sigma}) - \frac{1}{2}\sinh\alpha\,(\hat{\alpha}'-\hat{\alpha})\cdot\vec{\sigma}$$

$$(\hat{\alpha}'\cdot\vec{\sigma})(\hat{\alpha}\cdot\vec{\sigma}) = (\hat{\alpha}'\cdot\hat{\alpha}) + i\,(\hat{\alpha}'\times\hat{\alpha})\cdot\vec{\sigma}$$

$$\simeq 1 - i\,\Delta\theta\cdot\sigma_z$$



$$\Rightarrow K(\vec{\alpha}')\,K^{-1}(\vec{\alpha}) = 1 + i\,\sinh^2\frac{\alpha}{2}\,\Delta\theta\cdot\sigma_z - \frac{1}{2}\sinh\alpha\;\Delta\theta(\hat{e}_\theta\cdot\vec{\sigma})$$

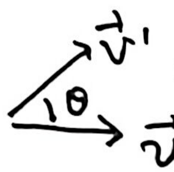
$$= \left(1 - \frac{1}{2}\sinh\alpha\cdot\Delta\theta\;\hat{e}_\theta\cdot\vec{\sigma}\right)\left(1 + i\,\frac{1}{2}\left[\cosh\alpha - 1\right]\Delta\theta\;\sigma_z\right]$$

$$K\left[\sinh\alpha\cdot\Delta\theta\;\hat{e}_\theta\right]\cdot U_{\hat{z}}\Big(-\underbrace{(\cosh\alpha-1)\cdot\Delta\theta}\Big)$$

this is a rotation around $\hat{z}$-axis

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \cosh\alpha \Rightarrow \cosh\alpha - 1 = \gamma - 1$$

Change to velocity



$$\delta\vec{\beta} = \frac{1}{c}(\vec{v}'-\vec{v})$$

$$\Delta\theta = \frac{\vec{v}\times\vec{v}'}{v^2} = \frac{\vec{\beta}\times(\vec{\beta}'-\vec{\beta})}{\beta^2}$$

$\gamma^2 - 1 = \beta^2\gamma^2$

Hence, the rotation angle $\Delta\Omega = -\dfrac{\gamma-1}{\beta^2}\,(\vec{\beta}\times\delta\vec{\beta})$

$$\Rightarrow \boxed{\Delta\Omega = -\frac{\gamma^2}{1+\gamma}\left(\vec{\beta}\times\delta\vec{\beta}\right)}$$

hence $\vec{\omega}_T = -\frac{\gamma^2}{1+\gamma}\left(\vec{\beta}\times\frac{d\vec{\beta}}{dt}\right)$

$$= -\frac{\gamma^2}{c^2(1+\gamma)}\left(\vec{v}\times\vec{a}\right) = \frac{\gamma^2}{1+\gamma}\ \frac{\vec{a}\times\vec{v}}{c^2}$$

neglect correction at β^2

already at v^2/c^2

we also neglect $\left(\frac{ds}{d\tau}\right)_M$ and $\left(\frac{ds}{dt}\right)_M$

For atomic problem

$$\omega_T \simeq \frac{1}{2c^2}\ \frac{F}{m^2}\ \frac{\vec{r}}{r}\times m\vec{v}$$

$$= -\frac{1}{2m^2c^2}\left(\frac{1}{r}\frac{dV}{dr}\right)\vec{L}$$

Hence $\boxed{U = -\frac{eg}{2mc}\ \vec{S}\cdot\vec{B} + \frac{(g-1)}{2m^2c^2}\left(\vec{S}\cdot\vec{L}\right)\left(\frac{1}{r}\frac{dV}{dr}\right)}$

Inside atom $\frac{dV}{dr}>0$, hence the low energy state

corresponds to $\underbrace{j_- = l-1/2}_{\text{the branch of}}$.

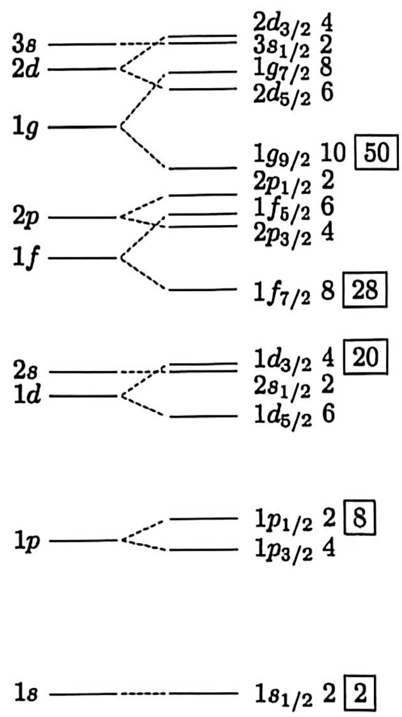
However, as for the problem of nuclear physics, $V_N(r)$ is not due to electric field. In fact, the EM part $V_{EM}(r)$ is repulsive, hence it contributes

$$\frac{g}{2m_n^2c^2}(\vec{S}\cdot\vec{L})\left(\frac{1}{r}\frac{dV_{EM}}{dr}\right) \rightarrow \frac{dV_{EM}}{dr}<0.$$

Thomas precession $\rightarrow$ $-\dfrac{1}{2m_n^2c^2}(\vec{S}\cdot\vec{L})\left(\dfrac{1}{r}\dfrac{d(V_N+V_{EM})}{dr}\right)$

$$\Rightarrow \quad U_{so} = -\frac{1}{2m_n^2c^2}(\vec{S}\cdot\vec{L})\left(\frac{1}{r}\right)\left[(1-g)\frac{dV_{EM}}{dr}+\frac{dV_N}{dr}\right]$$

dV_N/dr dominates



Relativistic Actions

We know in non-relativistic mechanics, the equation of motion can be derived via the least action principle

$$S = \int_{t_1}^{t_2} dt \; \mathcal{L}(x, \dot{x}), \quad \text{with } \mathcal{L} = \tfrac{1}{2} m \dot{x}^2 - V(x).$$

Let us first consider the case of free space, $V(x) = 0$.

$$\Rightarrow \quad S = \tfrac{1}{2} m \int dt \left(\frac{dx}{dt}\right)^2 = m/2 \int (dx)^2/dt \quad \leftarrow$$

dx, and dt are treated differently

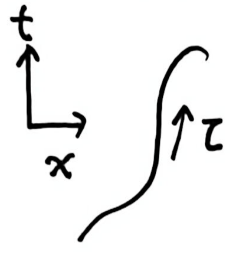
Remember we know $\dfrac{\varepsilon^2}{2a} \simeq -\sqrt{a^2 - \varepsilon^2} + a$

- then $S \simeq -mc \int \sqrt{(c\,dt)^2 - (dx)^2} + mc^2 \int dt$ ← drop away since it's variation $= 0$.

it's nice that $d\tau = \sqrt{(c\,dt)^2 - (dx)^2}$ is the proper time that measured in the comoving frame with the particle. It's invariant under Lorentz transformation. More symmetrically, we rewrite the relativistic action for a point particle as

$$S = -mc \int \sqrt{\eta_{\mu\nu}\, dx^\mu\, dx^\nu} \quad \text{where } \eta_{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

$$dx^\mu = (c\,dt, dx) \text{ and } dx_\mu = \eta_{\mu\nu}\, dx^\nu = (c\,dt, -dx)$$



$$(d\tau)^2 = (d\tau)^2 - (d\vec{x})^2$$

the "length" or proper time

reparameterization

$$S = -mc\int d\xi \sqrt{\eta_{\mu\nu}\frac{dx^\mu}{d\xi}\frac{dx^\nu}{d\xi}} = \int d\xi \cdot L$$

parameter along the curve

$$\delta S = 0 \Rightarrow \frac{d}{d\xi}\left(\frac{\delta L}{\delta(dx^\lambda/d\xi)}\right) = 0 \Rightarrow m\frac{d}{d\xi}\left(\frac{1}{L}\eta_{\mu\lambda}\frac{dx^\mu}{d\xi}\right) = 0$$

If take $d\xi = d\tau$, then $L = -m \Rightarrow \boxed{\frac{d^2}{d\tau^2}x^\mu = 0}$

- How to add potential $V(x)$?

We could try two different choices:

A: $S = -\left(\int mc\sqrt{\eta_{\mu\nu}dx^\mu dx^\nu} + V(x)dt\right)$

B: $S = -mc\int \sqrt{(1+\frac{V}{mc^2})(dt)^2 - (dx)^2}$

If we use the option A, it automatically yields $\frac{1}{2}m\dot{x}^2 - V(x)$.
If we interprete $V(x)$ as the electric potential energy
then $-V(x) = qA_0(x)\,dt$. To restore the Lorentz invariance, $qA_0(x)dt \to q/c\,(A_\mu(x)dx^\mu)$

$$\Rightarrow A: \; S = -mc\int \sqrt{\eta_{\mu\nu}dx^\mu dx^\nu} + \frac{q}{c}A_\mu(x)\,dx^\mu$$

This will lead to E&M.

parameterize with proper time $d\tau$

$$S = -mc\int d\tau \sqrt{\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}} + \int d\tau \; A_\mu(x(\tau))\frac{dx^\mu}{d\tau} \cdot q/c$$

$$\delta\left(-mc\int d\tau \sqrt{\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}\right) = -mc\int d\tau \; \frac{\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{d\delta x^\nu}{d\tau}}{\sqrt{\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}}$$

$$\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = \left[(c\,dt)^2-(dx)^2\right]/d\tau^2 = c$$

$$\Rightarrow \delta(\cdots) = -m\int d\tau \; \eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{d\delta x^\nu}{d\tau}$$

$$= m\int d\tau \; \eta_{\mu\nu}\frac{d^2x^\mu}{d\tau^2}\,\delta x^\nu$$

$$\delta\int d\tau \; A_\mu(x)\frac{dx^\mu}{d\tau}\frac{q}{c} = \int d\tau \left\{A_\mu(x)\frac{d\delta x^\mu}{d\tau} + [\partial_\nu A_\mu]\,\delta x^\nu \frac{dx^\mu}{d\tau}\right\} \cdot q/c$$

$$= \qquad - d\,A_\mu(x)\,\delta x^\mu$$

$$= -\partial_\nu A_\mu \, dx^\nu \, \delta x^\mu$$

$$= \int d\tau \; [\underbrace{\partial_\mu A_\nu - \partial_\nu A_\mu}_{F_{\mu\nu}}]\frac{dx^\nu}{d\tau} \cdot \delta x^\mu$$

$$\Rightarrow \delta S = \int d\tau \left[\; m\,\eta_{\mu\nu}\frac{d^2x^\mu}{d\tau^2}\,\delta x^\nu + F_{\mu\nu}\frac{dx^\nu}{d\tau}\cdot\delta x^\mu\right]$$

$$m\,\eta_{\nu\mu'}\frac{d^2x^{\mu'}}{d\tau^2} = -\,F_{\nu\lambda}\frac{dx^\lambda}{d\tau} \quad \leftarrow \quad \eta^{\mu\nu}\eta_{\nu\mu'} = \delta^\mu{}_{\mu'}$$

$$\Rightarrow \boxed{m\, d^2x^\mu/d\tau^2 = -\,\eta^{\mu\nu}F_{\nu\lambda}\frac{dx^\lambda}{d\tau}}$$

plug in $d\tau = \sqrt{1-\beta^2}\, dt \qquad \beta = v/c$

$$\Rightarrow m\frac{d}{dt}\left[\frac{d}{d\tau}x^\mu\right] = -\eta^{\mu\nu} F_{\nu\lambda}\frac{dx^\lambda}{dt}\cdot q/c$$

$$\Rightarrow \frac{d}{dt}\left[\frac{m}{\sqrt{1-\beta^2}}\frac{d}{dt}x^\mu\right] = -\eta^{\mu\nu} F_{\nu\lambda}\frac{dx^\lambda}{dt}\, q/c$$

$A^\mu = (\vec{A}, \varphi)$

$\partial^\mu = (-\vec{\partial}, \frac{1}{c}\partial_t)$

$$F^{12} = \partial^1 A^2 - \partial^2 A^1 = B^3 = F_{12}$$

$$F^{23} = F_{23} = B^1, \quad F^{31} = F_{13} = B^2$$

$$E^1 = F^{01} = \partial^0 A^1 - \partial^1 A^0 = \frac{1}{c}\partial_t A_x - \partial_x \varphi = -F_{01}$$

$$E^2 = F^{02} = -F_{02}, \quad E^3 = F^{03} = -F_{03}$$

① $\mu = 1 \Rightarrow$

$$\frac{d}{dt}[\gamma m \dot{x}] = \underbrace{(F_{12}\dot{y} + F_{13}\dot{z})}_{q/c} + F_{10}q$$

$$= \frac{1}{c}(B_z\dot{y} - B_y\dot{z}) + qE_x$$

$$\Rightarrow \frac{d}{dt}[\gamma m\vec{v}] = \frac{d}{dt}\vec{p} = q\left(\frac{\vec{v}}{c}\times\vec{B} + \vec{E}\right)$$

② $\mu = 0$

$$\frac{d}{dt}\left[\frac{mc}{\sqrt{1-\beta^2}}\right] = -F_{0i}\frac{dx^i}{dt}\frac{q}{c}$$

$$= q\vec{E}\cdot\vec{v}/c$$

$$\boxed{\frac{d}{dt}\left[\frac{mc^2}{\sqrt{1-\beta^2}}\right] = q\vec{E}\cdot\vec{v}}$$

Choice B: $S = -mc\int\sqrt{(1+\frac{2V}{mc^2})c^2dt^2 - (dx)^2}$

$$S = -mc\int\left\{cdt\sqrt{1+\frac{2V}{mc^2}} - \frac{1}{2}\frac{(dx)^2}{\sqrt{1+2V/mc^2}\,cdt}\right\}$$

$$= -mc\int\left\{cdt\left[1+\frac{V}{mc^2}\right] - \frac{1}{2}\frac{(dx)^2}{cdt}\right\}$$

$$= -mc^2\int dt + \frac{1}{2}m\int\frac{(dx)^2}{dt} - \int V\,dt$$

drop $\frac{V}{mc^2}$-order term

This can be intepreted as $\begin{cases} g_{00} = (1+\frac{2V}{mc^2}) & g_{0i} = g_{i0} = 0 \\ g_{ij} = -\delta_{ij} \end{cases}$

For the gravity field $V = -GMm/r \Rightarrow S = -m\int\sqrt{(1-\frac{2GM}{r^2})dt^2 - dx^2}$

Hence $d\tau = (1-\frac{GM}{r})dt$: dt is the time interval in the absence of gravity

$\Delta\tau < \Delta t$ $\qquad$ $d\tau$: ... in the gravity field

A clock runs slow in the gravity field.

If we intepreto $1+\frac{2V}{mc^2}$ as the metric, then V/m should be the same for particles of different mass. — This is the feature of gravity! (Gravity mass = inertial mass)