Geodesic equation — curvature

• Analogous to gauge field strength

$$D_\mu \psi = \left(\partial_\mu + \frac{ie}{c\hbar} A_\mu\right)\psi \rightarrow [D_\mu, D_\nu] = (\partial_\mu A_\nu - \partial_\nu A_\mu)\frac{ie}{\hbar c}$$
$$= \frac{ie}{\hbar c} F_{\mu\nu}.$$

✱ covariant derivative over vector

$$D_\mu A^\rho = \partial_\mu A^\rho + \Gamma_\mu{}^\rho{}_\lambda A^\lambda$$

$$D_\nu(D_\mu A^\rho) = \partial_\nu(D_\mu A^\rho) - \Gamma^\lambda_{\nu\mu} D_\lambda A^\rho + \Gamma^\rho_{\nu\lambda} D_\mu A^\lambda$$

$$= \partial_\nu(\partial_\mu A^\rho + \Gamma^\rho_{\mu\lambda} A^\lambda) - \Gamma^\lambda_{\nu\mu}(\partial_\lambda A^\rho + \Gamma_\lambda{}^\rho{}_\sigma A^\sigma) + \Gamma^\rho_{\nu\lambda}(\partial_\mu A^\lambda + \Gamma^\lambda_{\mu\sigma} A^\sigma$$

$$= \partial_\nu\partial_\mu A^\rho + \Gamma^\rho_{\mu\lambda}\partial_\nu A^\lambda - \Gamma^\lambda_{\nu\mu}\partial_\lambda A^\rho + \Gamma^\rho_{\nu\lambda}\partial_\mu A^\lambda$$
$$+ \partial_\nu \Gamma^\rho_{\mu\lambda} A^\lambda + (-\Gamma^\lambda_{\nu\mu}\Gamma^\rho_{\lambda\sigma} + \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}) A^\sigma$$

exchange $\mu \leftrightarrow \nu$

$$D_\mu(D_\nu A^\rho) = \partial_\mu\partial_\nu A^\rho + \Gamma^\rho_{\nu\lambda}\partial_\mu A^\lambda - \Gamma^\lambda_{\nu\mu}\partial_\lambda A^\rho + \Gamma^\rho_{\mu\lambda}\partial_\nu A^\lambda$$
$$+ \partial_\mu \Gamma^\rho_{\nu\lambda} A^\lambda + (-\Gamma^\lambda_{\nu\mu}\Gamma^\rho_{\lambda\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma}) A^\sigma$$

$$[D_\mu, D_\nu] A^\rho = \left[\partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}\right] A^\sigma = R^\rho_{\sigma\mu\nu} A^\sigma$$

where we define

$$\boxed{R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\sigma\nu} - \partial_\nu \Gamma^\rho_{\sigma\mu} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\sigma\nu} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\sigma\mu}}$$

independent of "A" itself, only related to "metric"!

$$\Rightarrow \quad R^\rho_{\sigma\mu\nu} = - R^\rho_{\sigma\nu\mu}$$ anti-symmetric under the exchange $\mu \leftrightarrow \nu$.

② $D_\mu A_\rho = \partial_\mu A_\rho - \Gamma^\lambda_{\mu\rho} A_\lambda$

$$D_\nu(D_\mu A_\rho) = \partial_\nu(D_\mu A_\rho) - \Gamma^\lambda_{\nu\mu} D_\lambda A_\rho - \Gamma^\lambda_{\nu\rho} D_\mu A_\lambda$$

$$= \partial_\nu[\partial_\mu A_\rho - \Gamma^\lambda_{\mu\rho}A_\lambda] - \Gamma^\lambda_{\nu\mu}[\partial_\lambda A_\rho - \Gamma^\sigma_{\lambda\rho}A_\sigma] - \Gamma^\lambda_{\nu\rho}[\partial_\mu A_\lambda - \Gamma^\sigma_{\mu\lambda}A_\sigma]$$

$$= \partial_\nu \cancel{\partial_\mu} A_\rho - \cancel{\Gamma^\lambda_{\mu\rho}}\partial_\nu A_\lambda - \Gamma^\lambda_{\nu\mu}\cancel{\partial_\lambda} A_\rho - \Gamma^\lambda_{\nu\rho}\cancel{\partial_\mu} A_\lambda$$

$$- \partial_\nu \Gamma^\lambda_{\mu\rho} A_\lambda + (\cancel{\Gamma^\lambda_{\nu\mu}}\Gamma^\sigma_{\lambda\rho} + \Gamma^\lambda_{\nu\rho}\Gamma^\sigma_{\mu\lambda})A_\sigma$$

$$D_\mu D_\nu A_\rho = \partial_\mu\cancel{\partial_\nu}A_\rho - \cancel{\Gamma^\lambda_{\nu\rho}}\partial_\mu A_\lambda - \cancel{\Gamma^\lambda_{\mu\nu}}\partial_\lambda A_\rho - \cancel{\Gamma^\lambda_{\mu\rho}}\partial_\nu A_\lambda$$

$$- \partial_\mu\Gamma^\lambda_{\nu\rho}A_\lambda + (\cancel{\Gamma^\lambda_{\mu\nu}}\Gamma^\sigma_{\lambda\rho} + \Gamma^\lambda_{\mu\rho}\Gamma^\sigma_{\nu\lambda})A_\sigma$$

$$[D_\mu, D_\nu]A_\rho = [-\partial_\mu\Gamma^\sigma_{\nu\rho} + \partial_\nu\Gamma^\sigma_{\mu\rho} + \Gamma^\lambda_{\mu\rho}\Gamma^\sigma_{\nu\lambda} - \Gamma^\lambda_{\nu\rho}\Gamma^\sigma_{\mu\lambda}]A_\sigma$$

$$\boxed{[D_\mu, D_\nu]A_\rho = -R^\sigma{}_{\rho\mu\nu}A_\sigma}$$

☆ Covariant derivatives on a tensor

$$D_\mu D_\nu T^{\lambda\rho} = \partial_\mu(D_\nu T^{\lambda\rho}) - \Gamma^\sigma_{\mu\nu}\cancel{D_\sigma}T^{\lambda\rho} + \Gamma^\lambda_{\mu\sigma}D_\nu T^{\sigma\rho} + \Gamma_\mu{}^\rho{}_\sigma D_\nu T^{\lambda\sigma}$$

$$D_\nu D_\mu T^{\lambda\rho} = \partial_\nu(D_\mu T^{\lambda\rho}) - \Gamma^\sigma_{\nu\mu}\cancel{D_\sigma}T^{\lambda\rho} + \Gamma^\lambda_{\nu\sigma}D_\mu T^{\sigma\rho} + \Gamma^\rho_{\nu\sigma}D_\mu T^{\lambda\sigma}$$

$$[D_\mu,D_\nu]T^{\lambda\rho} = \partial_\mu[\partial_\nu T^{\lambda\rho} + \Gamma^\lambda_{\nu\sigma}T^{\sigma\rho} + \Gamma^\rho_{\nu\sigma}T^{\lambda\sigma}]$$

$$+ \Gamma^\lambda_{\mu\sigma}[\partial_\nu T^{\sigma\rho} + \Gamma^\sigma_{\nu\sigma'}T^{\sigma'\rho} + \Gamma^\rho_{\nu\sigma'}T^{\sigma\sigma'}]$$

$$+ \Gamma^\rho_{\mu\sigma}[\partial_\nu T^{\lambda\sigma} + \Gamma^\lambda_{\nu\sigma'}T^{\sigma'\sigma} + \Gamma^\sigma_{\nu\sigma'}T^{\lambda\sigma'}] - (\mu\leftrightarrow\nu)$$

$$= \partial_\mu\partial_\nu T^{\lambda\rho} + \partial_\mu\Gamma^\lambda_{\nu\sigma}T^{\sigma\rho} + \Gamma^\lambda_{\nu\sigma}\partial_\mu T^{\sigma\rho} + \partial_\mu\Gamma^\rho_{\nu\sigma}T^{\lambda\sigma} + \Gamma^\rho_{\nu\sigma}\partial_\mu T^{\lambda\sigma}$$

$$\Gamma^\lambda_{\mu\sigma}\partial_\nu T^{\sigma\rho} + \qquad + \Gamma^\rho_{\mu\sigma}\partial_\nu T^{\lambda\sigma}$$

$$+ \underline{\Gamma^\lambda_{\mu\sigma}\Gamma^\rho_{\nu\sigma'}T^{\sigma\sigma'} + \Gamma^\rho_{\mu\sigma}\Gamma^\lambda_{\nu\sigma'}T^{\sigma'\sigma}} + \Gamma^\lambda_{\mu\sigma}\Gamma^\sigma_{\nu\sigma'}T^{\sigma'\rho} + \Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\nu\sigma'}T^{\lambda\sigma'} - (\mu\leftrightarrow\nu)$$

$$[D_\mu, D_\nu] T^{\lambda\rho} = \left(\partial_\mu \Gamma^\lambda_{\nu\sigma} - \partial_\nu \Gamma^\lambda_{\mu\sigma} + \Gamma^\lambda_{\mu\sigma'} \Gamma^{\sigma'}_{\nu\sigma} - \Gamma^\lambda_{\nu\sigma'} \Gamma^{\sigma'}_{\nu\sigma}\right) T^{\sigma\rho}$$
$$+ \left(\partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\sigma'} \Gamma^{\sigma'}_{\nu\sigma} - \Gamma^\rho_{\nu\sigma'} \Gamma^{\sigma'}_{\mu\sigma}\right) T^{\lambda\sigma}$$

$$\boxed{[D_\mu, D_\nu] T^{\lambda\rho} = R^\lambda_{\ \sigma\mu\nu} T^{\sigma\rho} + R^\rho_{\ \sigma\mu\nu} T^{\lambda\sigma}}$$

(*) $$\boxed{R^\rho_{\ \lambda\mu\nu} + R^\rho_{\ \mu\nu\lambda} + R^\rho_{\ \nu\lambda\mu} = 0}$$

Proof: $R^\rho_{\ \lambda\mu\nu} = \partial_\mu \Gamma^\rho_{\lambda\nu} - \partial_\nu \Gamma^\rho_{\lambda\mu} + \Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\lambda\nu} + \Gamma^\rho_{\nu\sigma}\Gamma^\sigma_{\lambda\mu}$

$R^\rho_{\ \mu\nu\lambda} = \partial_\nu \Gamma^\rho_{\mu\lambda} - \partial_\lambda \Gamma^\rho_{\mu\nu} + \Gamma^\rho_{\nu\sigma}\Gamma^\sigma_{\mu\lambda} + \Gamma^\rho_{\lambda\sigma}\Gamma^\sigma_{\mu\nu}$

$R^\rho_{\ \nu\lambda\mu} = \partial_\lambda \Gamma^\rho_{\nu\mu} - \partial_\mu \Gamma^\rho_{\nu\lambda} + \Gamma^\rho_{\lambda\sigma}\Gamma^\sigma_{\nu\mu} + \Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\nu\lambda}$

Add together $= 0$

Example: consider $d\tau^2 = \frac{1}{t^2}(dt^2 - dx^2)$

$g_{tt} = 1/t^2$, $g_{xx} = -1/t^2$, $g^{tt} = t^2$, $g^{xx} = -t^2$

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\lambda}(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu})$$

$$\Gamma^t_{tt} = \frac{1}{2} g^{t\lambda}(\partial_t g_{t\lambda} + \partial_t g_{t\lambda} - \partial_\lambda g_{tt}) = \frac{1}{2} g^{tt}\partial_t g_{tt} = \frac{t^2}{2}\partial_t t^{-2} = -1/t$$

$$\Gamma^t_{xx} = \frac{1}{2} g^{t\lambda}(\partial_x g_{x\lambda} + \partial_x g_{x\lambda} - \partial_\lambda g_{xx}) = -\frac{1}{2} g^{tt}\partial_t g_{xx} = -\frac{1}{2}t^2\partial_t(-1/t^2) = -1/t$$

$$\Gamma^x_{tx} = \Gamma^x_{xt} = \frac{1}{2} g^{x\lambda}(\partial_x g_{t\lambda} + \partial_t g_{x\lambda} - \partial_\lambda g_{xt}) = \frac{1}{2}\left[g^{xx}\partial_t g_{xx}\right] = -\frac{t^2}{2}\partial_t\left(-\frac{1}{t^2}\right) = -1/t$$

Geodesic equation

$$\frac{d^2x^\ell}{ds^2} + \Gamma^\ell_{kj}\frac{dx^k}{ds}\frac{dx^j}{ds} = 0$$

$$\frac{d^2t}{ds^2} + \Gamma^t_{tt}\left(\frac{dt}{ds}\right)^2 + \Gamma^t_{xx}\left(\frac{dx}{ds}\right)^2 = 0 \Rightarrow \frac{d^2t}{ds^2} - 1/t\left[\left(\frac{dt}{ds}\right)^2 + \left(\frac{dx}{ds}\right)^2\right] = 0$$

$$\frac{d^2x}{ds^2} + 2\Gamma^x_{xt}\frac{dx}{ds}\frac{dt}{ds} = 0 \Rightarrow \frac{d^2x}{ds^2} - \frac{2}{t}\frac{dx}{ds}\frac{dt}{ds} = 0.$$

$$R^t_{xtx} = \partial_t \Gamma^t_{xx} - \partial_x \Gamma^t_{xt} + \Gamma^t_{t\lambda}\Gamma^\lambda_{xx} - \Gamma^t_{x\lambda}\Gamma^\lambda_{xt}$$

$$= \partial_t[-1/t] + \Gamma^t_{tt}\Gamma^t_{xx} - \Gamma^t_{xx}\Gamma^x_{xt} = 1/t^2 + (-1/t)^2 - (-1/t)(-1/t)^2 = 1/t^2$$

$$R_{txtx} = g_{tt}R^t_{xtx} = 1/t^2 \cdot 1/t^2 = 1/t^4.$$

(*) Parallel transport on a sphere

Consider a vecter $A^\alpha = (A^\theta, A^\varphi)$. On a sphere $ds^2 = d\theta^2 + \sin^2\theta d\varphi^2$

$$g_{\theta\theta} = 1, \quad g_{\phi\phi} = \sin^2\theta, \Rightarrow g^{\theta\theta} = 1, \quad g^{\phi\phi} = \frac{1}{\sin^2\theta}$$

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2}g^{\sigma\lambda}(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu})$$

$$\Gamma^\theta_{\phi\phi} = \frac{1}{2}g^{\theta\theta}(\partial_\phi g_{\phi\theta} + \partial_\phi g_{\phi\theta} - \partial_\theta g_{\phi\phi}) = -\frac{1}{2}\partial_\theta \sin^2\theta = -\sin\theta\cos\theta$$

$$\Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \frac{1}{2}g^{\phi\phi}(\partial_\theta g_{\phi\phi} + \partial_\phi g_{\theta\phi} - \partial_\phi g_{\theta\phi}) = \frac{1}{2}\frac{1}{\sin^2\theta}\partial_\theta \sin^2\theta$$

$$= \cot\theta.$$

Consider transport along $x(\tau) \Rightarrow \frac{dx^\nu}{d\tau}(\partial_\nu A^\mu + \Gamma^\mu_{\nu\lambda}A^\lambda) = 0$

Along $\theta = \theta_0$, φ is the variable

$$\partial_\varphi A^\mu + \Gamma^\mu_{\varphi\lambda}A^\lambda = 0 \Rightarrow \begin{cases} \partial_\phi A^\theta + \Gamma^\theta_{\phi\phi}(\theta=\theta_0)A^\phi = 0, \\ \partial_\phi A^\varphi + \Gamma^\phi_{\phi\theta}(\theta=\theta_0)A^\theta = 0. \end{cases}$$

$$\Rightarrow \begin{cases} \partial_\phi A^\theta = \frac{1}{2}\sin 2\theta_0 A^\varphi \\ \partial_\phi A^\phi = -\cot\theta_0 A^\theta \end{cases} \Rightarrow \partial^2_\phi A^\phi = -\cos^2\theta_0 A^\phi$$

If imposing the initial condition $A_\theta^{\theta=\theta_0} = 1$, $A^{\phi=0} = 0$, i.e. A along A^θ

$$\Rightarrow A^\phi = C\sin(\phi\cos\theta_0)$$

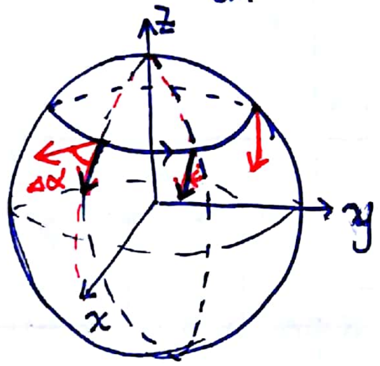
$$A^\theta = -\tan\theta_0\, \partial_\phi A^\varphi = -C\tan\theta_0 \cos(\phi\cos\theta_0)\cos\theta_0 = -C\sin\theta_0 \cos(\phi\cos\theta_0)$$

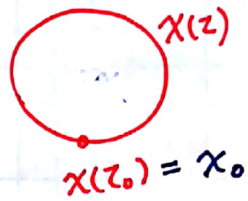
$$\Rightarrow \boxed{\begin{cases} A^\theta = \cos(\phi\cos\theta_0) \\ A^\varphi = -\csc\theta_0 \sin(\phi\cos\theta_0) \end{cases}} \Rightarrow \begin{aligned} & g_{\theta\theta} A^{\theta^2} + g_{\varphi\varphi} A^{\varphi^2} \\ &= \cos^2(\phi\cos\theta_0) + \sin^2(\phi\cos\theta_0) = 1 \end{aligned}$$

However, when the vector goes round θ_0, $\rightarrow \phi = 2\pi$

$$\begin{cases} A^\theta = -\cos(2\pi\cos\theta_0) \\ A^\varphi = -\csc\theta_0 \underset{\sin}{(2\pi\cos\theta_0)} \end{cases} \longrightarrow \vec{A} = \cos(2\pi\cos\theta_0)\,\hat{e}_\theta - \sin(2\pi\cos\theta_0)\,\hat{e}_\varphi$$

$$\boxed{\Delta\alpha = -2\pi\cos\theta_0}$$





(*) Around an infinitsmall loop, the parallel transport

$$d\xi^\mu = -\Gamma^\mu_{\nu\lambda}\, dx^\nu\, \xi^\lambda$$

$$\Delta\xi^\mu = \oint d\xi^\mu = \oint d\tau \frac{d\xi^\mu}{d\tau} = -\oint d\tau\, \Gamma^\mu_{\nu\lambda}(x) \frac{dx^\nu}{d\tau} \xi^\lambda(x)$$

$$\Gamma^\mu_{\nu\lambda}(x) = \Gamma^\mu_{\nu\lambda}(x_0) + (x-x_0)^\rho \partial_\rho \Gamma^\mu_{\nu\lambda}(x_0)$$

$$\xi^\mu(x) \simeq \xi^\mu(x_0) - \Gamma^\mu_{\nu\lambda}(x_0)(x-x_0)^\nu \xi^\lambda(x_0) \leftarrow \text{parallel}$$

$$\Delta\xi^\mu = \oint d\tau \, -\left[\Gamma^\mu_{\nu\lambda}(x_0) + (x-x_0)^\gamma \partial_\gamma \Gamma^\mu_{\nu\lambda}(x_0) + \cdots\right] \left[\xi^\lambda(x_0) - \Gamma^\lambda_{\rho\sigma}(x-x_0)^\rho \xi^\sigma(x_0)\right] \frac{dx^\nu}{d\tau}$$

$\frac{dx^\nu}{d\tau}$

$$= \oint d\tau \, -\Gamma^\mu_{\nu\lambda}(x_0)\xi^\lambda(x_0)\frac{dx^\nu}{d\tau} + (x-x_0)^\rho \xi^\sigma(x_0)\{-\partial_\rho \Gamma^\mu_{\nu\sigma}(x_0) + \Gamma^\mu_{\nu\lambda}(x_0)\Gamma^\lambda_{\rho\sigma}(x_0)$$

For closed loop $\oint \frac{dx^\nu}{d\tau} = 0$.

$$\Delta\xi^\mu = \xi^\sigma(x_0)\{-\partial_\rho \Gamma^\mu_{\nu\sigma}(x_0) + \Gamma^\mu_{\nu\lambda}(x_0)\Gamma^\lambda_{\rho\sigma}(x_0)\}\oint d\tau\,(x-x_0)^\rho \frac{dx^\nu}{d\tau}$$

$$\oint d\tau\, x^\rho \frac{dx^\nu}{d\tau} = -\oint d\tau\, x^\nu \frac{dx^\rho}{d\tau}$$

$$\Rightarrow \Delta\xi^\mu = \frac{1}{2}\xi^\sigma(x_0)\left[-\partial_\rho\Gamma^\mu_{\nu\sigma}(x_0) + \partial_\nu\Gamma^\mu_{\rho\sigma}(x_0) + \Gamma^\mu_{\nu\lambda}(x_0)\Gamma^\lambda_{\rho\sigma}(x_0) - \Gamma^\mu_{\rho\lambda}(x_0)\Gamma^\lambda_{\nu\sigma}(x_0)\right]$$

$$\oint d\tau\, x^\rho dx^\nu/d\tau$$

$$= \frac{1}{2}\xi^\sigma(x_0)\, R^\mu_{\ \sigma\nu\rho}(x_0)\, \underbrace{\oint d\tau\, x^\rho \frac{dx^\nu}{d\tau}}_{} \leftarrow \text{area of the loop}$$

Hence, the change of a vector after parallel transport around a loop $\propto$ the curvature tensor and the area of the loop.

$$\boxed{\Delta\xi^\mu = \frac{1}{2} R^\mu_{\ \sigma\nu\rho}(x_0)\,\xi^\sigma A^{\nu\rho} \leftarrow A^{\nu\rho} = \oint dx^\nu x^\rho}$$

① If the curvatures vanish, then $\Delta\xi^\mu = 0. \Rightarrow D_\nu\xi^\mu = 0$

$\Delta\xi^\mu = 0$ means that the parallel transport does not depend on the path. The we define $\xi^\mu(x)$ as the value of $\xi^\mu(x_0)$ being parallelly transported to x. The we have $\frac{d\xi^\mu}{d\tau} = \frac{dx^\nu}{d\tau}\frac{\partial\xi^\mu}{\partial x^\nu}$, we also have $\frac{d\xi^\mu}{d\tau} = -\Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\tau}\xi^\lambda$, hence, by equalling these two

$$D_\lambda\xi^\mu = \partial_\nu\xi^\mu + \Gamma^\mu_{\nu\lambda}\xi^\lambda = 0.$$

2) Conversely, if there exist a vector field, \xi^\lambda, such that D_mu xi^\lambda=0, then [D_\mu D_\nv] \xi^\lambda=0, => R^\lambda_{\sigma, \mu, \nv}=0. Hence, locally it is flat.

Geodesic equation:

In a locally flat Cartesian coordinate, the geodesic Eq is $d^2x^\mu/d\tau^2 = 0$, where $d\tau^2 = \eta_{\mu\nu}\, dx^\mu dx^\nu$ - propertime.

Now let's transform to a general coordinate frame

$x^\mu(\tau) \to x'^\mu(x(\tau))$ then $\frac{dx^\mu}{d\tau} \to \frac{dx'^\mu}{d\tau} = \frac{\partial x'^\mu}{\partial x^\nu}\frac{dx^\nu}{d\tau}$

$$\frac{d^2x^\mu}{d\tau^2} \to \frac{d}{d\tau}\left(\frac{dx'^\mu}{d\tau}\right) = \frac{d}{d\tau}\left(\frac{\partial x'^\mu}{\partial x^\nu}\frac{dx^\nu}{d\tau}\right)$$

$$= \frac{\partial x'^\mu}{\partial x^\nu}\frac{d^2x^\nu}{d\tau^2} + \frac{\partial^2 x'^\mu}{\partial x^\lambda \partial x^\nu}\frac{dx^\lambda}{d\tau}\frac{dx^\nu}{d\tau}$$

Further $\Gamma^\mu_{\nu\lambda} \to \Gamma'^\mu_{\nu\lambda} = \frac{\partial x'^\mu}{\partial x^{\mu_1}}\frac{\partial x^{\nu_1}}{\partial x'^\nu}\frac{\partial x^{\lambda_1}}{\partial x'^\lambda}\Gamma^{\mu_1}_{\nu_1\lambda_1} - \frac{\partial^2 x'^\mu}{\partial x^{\nu_1}\partial x^{\lambda_1}}\frac{\partial x^{\nu_1}}{\partial x'^\nu}\frac{\partial x^{\lambda_1}}{\partial x'^\lambda}$

So that

$$\Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} \to \Gamma'^\mu_{\nu\lambda}\frac{dx'^\nu}{d\tau}\frac{dx'^\lambda}{d\tau} = \left[\frac{\partial x'^\mu}{\partial x^{\mu_1}}\frac{\partial x^{\nu_1}}{\partial x'^\nu}\frac{\partial x^{\lambda_1}}{\partial x'^\lambda}\Gamma^{\mu_1}_{\nu_1\lambda_1} + \frac{\partial^2 x'^\mu}{\partial x^{\nu_1}\partial x^{\lambda_1}}\frac{\partial x^{\nu_1}}{\partial x'^\nu}\frac{\partial x^{\lambda_1}}{\partial x'^\lambda}\right]$$

$$\cdot \frac{\partial x'^\nu}{\partial x^{\nu_2}}\frac{dx^{\nu_2}}{d\tau} \cdot \frac{\partial x'^\lambda}{\partial x^{\lambda_2}}\frac{dx^{\lambda_2}}{d\tau}$$

$$= \delta^{\nu_1}_{\nu_2}\delta^{\lambda_1}_{\lambda_2}\left[\frac{\partial x'^\mu}{\partial x^{\mu_1}}\Gamma^{\mu_1}_{\nu_1\lambda_1} - \frac{\partial^2 x'^\mu}{\partial x^{\nu_1}\partial x^{\lambda_1}}\right]\frac{dx^{\nu_2}}{d\tau}\frac{dx^{\lambda_2}}{d\tau}.$$

$$= \left[\frac{\partial x'^\mu}{\partial x^{\mu_1}}\Gamma^{\mu_1}_{\nu_1\lambda_1} - \frac{\partial^2 x'^\mu}{\partial x^{\nu_1}\partial x^{\lambda_1}}\right]\frac{dx^{\nu_1}}{d\tau}\frac{dx^{\lambda_1}}{d\tau}.$$

$$\Rightarrow \frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} \to \frac{d^2x'^\mu}{d\tau^2} - \Gamma'^\mu_{\nu\lambda}\frac{dx'^\nu}{d\tau}\frac{dx'^\lambda}{d\tau} = \frac{\partial x'^\mu}{\partial x^\sigma}\left[\frac{d^2x^\sigma}{d\tau^2} + \Gamma^\sigma_{\nu\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau}\right]$$

$$= 0$$

Then we hope it works for curved space

we define 4-velocity $u^\mu = dx^\mu/d\tau$. In flat space,

$$du^\mu/d\tau = 0$$

In a curved space, we replace $\frac{d}{d\tau} \to D_\tau$

$$Du^\mu/d\tau = 0 \Rightarrow \frac{d}{d\tau} u^\mu - \Gamma^\mu_{\nu\lambda} \frac{dx^\nu}{d\tau} u^\lambda = 0.$$

$$Du^\mu = du^\mu - \Gamma^\mu_{\nu\lambda} dx^\nu u^\lambda$$

④ Geometric meaning of parallel transport

— • generalized "straightest line" in a curved manifold.

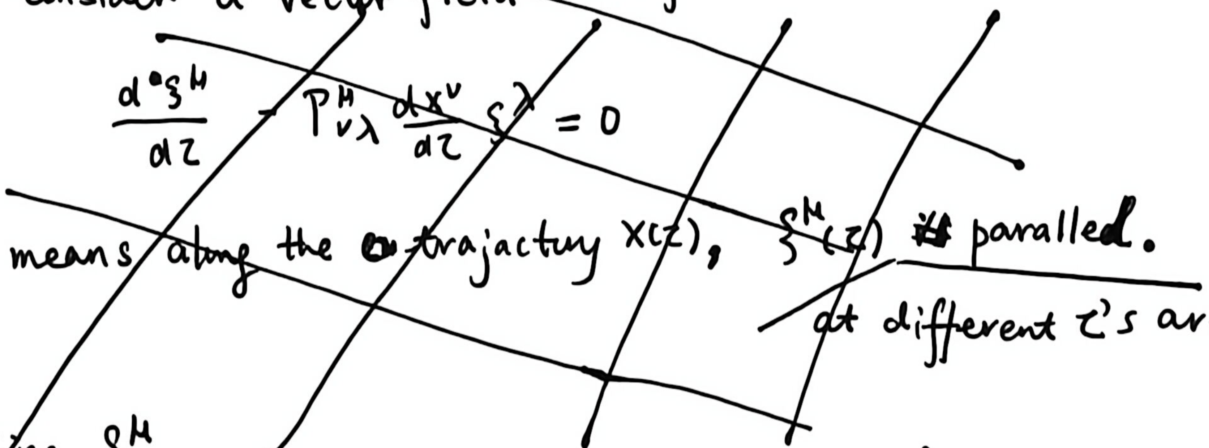
For a parallel transport of a vector $\xi^\mu(x(\tau))$ along $x(\tau)$

$$\rightarrow \quad \frac{d\xi^\mu}{d\tau} = \Gamma^\mu_{\nu\lambda} \frac{dx^\nu}{d\tau} \xi^\lambda .$$

Now consider a vector field satisfies the Eq

$$\frac{d\xi^\mu}{d\tau} - \Gamma^\mu_{\nu\lambda} \frac{dx^\nu}{d\tau} \xi^\lambda = 0$$

This means along the trajectory $x(\tau)$, $\xi^\mu(\tau)$ at different τ's are parallel.



Define ξ^μ

We choose $\xi^\mu = \frac{dx^\mu}{d\tau}$ the tangent vector, and parallelly transport it,

$$\Rightarrow \quad \frac{d^2 x^\mu}{d\tau^2} - \Gamma^\mu_{\nu\lambda} \frac{dx^\nu}{d\tau} \frac{dx^\lambda}{d\tau} = 0.$$

Hence, the tangent vectors at different τ's are parallel to each other

Solution of a geodesic Eq: plane in polar coordinate

$$d\tau^2 = dr^2 + r^2 d\theta^2$$

$$\Rightarrow g_{rr} = 1, \quad g_{\theta\theta} = r^2, \quad g^{rr} = 1, \quad g^{\theta\theta} = 1/r^2$$

$$\Gamma^r_{\theta\theta} = -r, \qquad \Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = 1/r$$

Geodesic $\quad \dfrac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda}\dfrac{dx^\nu}{d\tau}\dfrac{dx^\lambda}{d\tau} = 0, \quad \mu, \nu, \lambda = r, \theta$

$$\frac{d^2 r}{d\tau^2} + \Gamma^r_{\theta\theta}\left(\frac{d\theta}{d\tau}\right)^2 = 0 \Rightarrow \frac{d^2r}{d\tau^2} - r\left(\frac{d\theta}{d\tau}\right)^2 = 0 \qquad (*)$$

$$\frac{d^2\theta}{d\tau^2} + 2\Gamma^\theta_{\theta r}\frac{d\theta}{d\tau}\cdot\frac{dr}{d\tau} = 0 \Rightarrow \frac{d^2\theta}{d\tau^2} + \frac{2}{r}\frac{d\theta}{d\tau}\frac{dr}{d\tau} = 0 \qquad (**)$$

$$(**) \Rightarrow \quad \rightarrow \left(\frac{d\theta}{d\tau}\right)^{-1}\frac{d}{d\tau}\left(\frac{d\theta}{d\tau}\right) + \frac{2}{r}\frac{dr}{d\tau} = 0$$

$$\frac{d}{d\tau}\left[\ln\left(\frac{d\theta}{d\tau}\, r^2\right)\right] = 0 \Rightarrow \boxed{r^2\frac{d\theta}{d\tau} = l = \text{const}}$$

$$(*) \Rightarrow \frac{d^2r}{d\tau^2} - r\left(\frac{d\theta}{dr}\right)^2 = 0 \Rightarrow 2\frac{dr}{d\tau}\frac{d}{d\tau}\left(\frac{dr}{d\tau}\right) - 2r\frac{dr}{d\tau}\left(\frac{d\theta}{d\tau}\right)^2 = 0$$

It can be proved $\quad \dfrac{d}{d\tau}\left(r^2\left(\dfrac{d\theta}{d\tau}\right)^2\right) = 2r^2\dfrac{d\theta}{d\tau}\dfrac{d^2\theta}{d\tau^2} + 2r\dfrac{dr}{d\tau}\left(\dfrac{d\theta}{d\tau}\right)^2$

$$= -\frac{4r^2}{r}\left(\frac{d\theta}{d\tau}\right)^2\frac{dr}{d\tau} + 2r\frac{dr}{d\tau}\left(\frac{d\theta}{d\tau}\right)^2$$

$$= -2r\frac{dr}{d\tau}\left(\frac{d\theta}{d\tau}\right)^2$$

$$\Rightarrow \quad \frac{d}{d\tau}\left(\frac{dr}{d\tau}\right)^2 - 2r\frac{dr}{d\tau}\left(\frac{d\theta}{d\tau}\right)^2 = 0$$

or $$\frac{d}{d\tau}\left(\frac{dr}{d\tau}\right)^2 + \frac{d}{d\tau}\left(r^2\left(\frac{d\theta}{d\tau}\right)^2\right) = 0$$

$$\Rightarrow \left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\theta}{d\tau}\right)^2 = \text{cnst} = 1.$$

we choose cnst $= 1$ means $d\tau$ is the line segement

then $\left.\begin{array}{l} r = r(\tau) \\ \theta = \theta(\tau) \end{array}\right\} \Rightarrow r = r(\theta)$

$$\frac{dr}{d\tau} = \frac{dr}{d\theta}\cdot\frac{d\theta}{d\tau}$$

$$d^2r/d\tau^2 = \left[\frac{d^2r}{d\theta^2} - \frac{2}{r}\left(\frac{dr}{d\theta}\right)^2\right]\left(\frac{d\theta}{d\tau}\right)^2 = r^2\left(\frac{d\theta}{d\tau}\right)^2$$

$$\Rightarrow \frac{d^2r}{d\theta^2} - \frac{2}{r}\left(\frac{dr}{d\theta}\right)^2 - r = 0, \qquad \text{(complicated)}$$

we know the solution $y = ax + b$

$$\Rightarrow r = \frac{b}{\sin\theta - a\cos\theta} \qquad \Rightarrow \frac{dr}{d\theta} = -b\,\frac{\cos\theta + a\sin\theta}{(\sin\theta - a\cos\theta)^2}$$

$$d^2r/d\theta^2 = r + \frac{2}{r}\left(\frac{dr}{d\theta}\right)^2.$$

Affine variable $s(\tau)$

$$\Rightarrow \frac{d}{d\tau}\left(\frac{ds}{d\tau}\frac{dx^\mu}{ds}\right) + \Gamma^\mu_{\nu\lambda}\left(\frac{ds}{d\tau}\frac{dx^\nu}{ds}\right)\left(\frac{ds}{d\tau}\frac{dx^\lambda}{d\tau}\right) = 0$$

$$d^2x^\mu/ds^2 + \Gamma^\mu_{\nu\lambda}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = -\frac{d^2s/d\tau^2}{\left(\frac{ds}{d\tau}\right)^2}\frac{dx^\mu}{ds}$$

hence, if $s = a\tau + b$, then geodesic Eq is invariant.

Action: $$S = m\int d\lambda\left(g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\right)^{1/2} = m\int d\tau$$

which works for $g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} > 0$, i.e. time-like.

$$\frac{\partial \mathcal{L}}{\partial \dot x^\mu} = \frac{1}{2}\frac{1}{(g_{\mu\nu}\dot x^\mu \dot x^\nu)^{1/2}}\, 2 g_{\mu\nu}\dot x^\nu$$

$$= \frac{1}{(g_{\mu\nu}\dot x^\mu \dot x^\nu)^{1/2}}\, g_{\mu\nu}\dot x^\nu \qquad d\tau = d\lambda\,(g_{\mu\nu}\dot x^\mu \dot x^\nu$$

$$= \frac{d\lambda}{d\tau}\, g_{\mu\nu}\frac{dx^\nu}{d\lambda} = g_{\mu\nu}\frac{dx^\nu}{d\tau}$$

$$\frac{\partial \mathcal{L}}{\partial x^\mu} = \frac{1}{2}\frac{1}{(g_{\alpha\beta}\dot x^\alpha \dot x^\beta)^{1/2}}\,\partial_\mu g_{\sigma\nu}\dot x^\sigma \dot x^\nu$$

$$= \frac{1}{2}\frac{d\lambda}{d\tau}\,\partial_\mu g_{\sigma\nu}\frac{dx^\sigma}{d\lambda}\frac{dx^\nu}{d\lambda} = \frac{1}{2}\frac{d\tau}{d\lambda}\,\partial_\mu g_{\sigma\nu}\frac{dx^\sigma}{d\tau}\frac{dx^\nu}{d\tau}$$

$$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}^\mu}\right) - \frac{\partial L}{\partial x^\mu} = 0$$

$$\Rightarrow \frac{d\tau}{d\lambda}\,\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x}^\mu}\right) - \frac{\partial L}{\partial x^\mu} = 0$$

$$\frac{d\tau}{d\lambda}\,\frac{d}{d\tau}\left(g_{\mu\nu}\frac{dx^\nu}{d\tau}\right) - \frac{1}{2}\frac{d\tau}{d\lambda}\,\partial_\mu g_{\sigma\nu}\frac{dx^\sigma}{d\tau}\frac{dx^\nu}{d\tau} = 0$$

$$\frac{d\tau}{d\lambda}\left[g_{\mu\nu}\frac{d^2x^\nu}{d\tau^2} + \partial_\sigma g_{\mu\nu}\frac{dx^\sigma}{d\tau}\frac{dx^\nu}{d\tau} - \frac{1}{2}\partial_\mu g_{\sigma\nu}\frac{dx^\sigma}{d\tau}\frac{dx^\nu}{d\tau}\right] = 0$$

$$g_{\mu\nu}\frac{d^2x^\nu}{d\tau^2} + \frac{1}{2}\left(\partial_\sigma g_{\mu\nu} + \partial_\nu g_{\mu\sigma} - \partial_\mu g_{\sigma\nu}\right)\frac{dx^\sigma}{d\tau}\frac{dx^\nu}{d\tau} = 0$$

$$\frac{d^2x^\lambda}{d\tau^2} + \frac{1}{2}g^{\lambda\mu}\left(\partial_\sigma g_{\mu\nu} + \partial g_{\mu\sigma} - \partial_\mu g_{\sigma\nu}\right)\frac{dx^\sigma}{d\tau}\frac{dx^\nu}{d\tau} = 0$$

$$\boxed{\frac{d^2x^\lambda}{d\tau^2} + \Gamma^\lambda_{\sigma\nu}\frac{dx^\sigma}{d\tau}\frac{dx^\nu}{d\tau} = 0}$$