Lecture 7 Geodecis and more

- Action of geodesics
- Gravitational red shift
- Twin paradox consistency
- Equation of motion in the presence of other forces.

§ Alternative action for the geodesic

The action $S = m\int d\tau \left(g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \right)^{1/2}$ only works for time-like domains. To avoid this difficulty, we introduce $\tilde{S} = \int d\tau \, \mathcal{L} = \frac{1}{2}\int d\tau \left(\frac{1}{\sqrt{F}} g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} + \sqrt{F} m^2 \right)$

F is an auxillary field.

$$\delta \tilde{S} / \delta F = 0 \Rightarrow \cancel{g_{\mu\nu}} d \quad \frac{1}{\sqrt{F}} g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = \sqrt{F} m^2$$

$$F = \frac{1}{m^2} g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \rightarrow \text{plug in } \tilde{S}$$

$$\Rightarrow \tilde{S} = m \int d\tau \sqrt{g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}}$$

§ Geodesic reproducing Newton's 2nd law

$$\phi(x) = -G_N M / |x|$$

$$dx^2/dt^2 = -\nabla \phi(\vec{x})$$

we want the geodesics reproducing Newton's 2nd law

In the weak field limit, we have $g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x)$ and $g^{\mu\nu}(x) = \eta^{\mu\nu} - h^{\mu\nu}(x)$ such that $g^{\mu\nu} g_{\nu\lambda} \simeq \delta^\mu_\lambda - h^\mu{}_\lambda + h^\mu{}_\lambda = \delta^\mu_\lambda$

we are considering a static problem, $\partial_0 g_{\mu\nu} = \partial_0 h_{\mu\nu} = 0$. ②

Then in the non-relativistic limit $\frac{dx^0}{d\tau} = \frac{dt}{d\tau} \sim 1 \gg \frac{dx^i}{d\tau}$

then
$$d^2x^\mu / d\tau^2 + \Gamma^\mu_{\nu\lambda} \frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} = 0$$
$$d^2x^\mu / d\tau^2 + \Gamma^\mu_{00} \frac{dt}{d\tau}\frac{dt}{d\tau} = 0$$

$$\Gamma^\mu_{\nu\lambda} = \tfrac{1}{2} g^{\mu\rho}(\partial_\nu g_{\lambda\rho} + \partial_\lambda g_{\nu\rho} - \partial_\rho g_{\nu\lambda})$$

$$\Rightarrow \Gamma^\mu_{00} = \tfrac{1}{2} g^{\mu\rho}(\partial_0 g_{0\rho} + \partial_0 g_{0\rho} - \partial_\rho g_{00}) = -\tfrac{1}{2} g^{\mu\rho}\partial_\rho g_{00}$$

$(x^0, x^i) = (t, x)$

$(\partial^0\ \partial^i) = (-\partial_t, +\partial x)$

$$\simeq -\tfrac{1}{2}\eta^{\mu\rho}\partial_\rho h_{00}$$

$$\Gamma^0_{00} = 0, \quad \Gamma^i_{00} = -\tfrac{1}{2}\eta^{ii}\partial_i h_{00} = -\tfrac{1}{2}\partial^i h_{00} = -\tfrac{1}{2}\nabla_i h_{00}$$

$$\Rightarrow d^2t/d\tau^2 = 0 \Rightarrow \frac{dt}{d\tau} = \text{const} \Rightarrow \tau = kt$$

$$\frac{d^2\vec{x}}{c^2 d\tau^2} - \frac{1}{2}\vec{\nabla} h_{00}(x)\left(\frac{dt}{d\tau}\right)^2 = 0$$

$$\frac{d^2\vec{x}}{c^2 dt^2} - \frac{1}{2}\vec{\nabla} h_{00}(x) = 0 \Rightarrow \frac{d^2\vec{x}}{c^2 dt^2} = \frac{1}{c^2}\nabla\frac{G_N M}{|x|} = -\frac{\nabla\phi}{c^2}$$

$$\Rightarrow \frac{1}{2}\nabla h_{00}(x) = -\frac{\nabla\phi}{c^2} \Rightarrow h_{00} = -2\phi(x)/c^2$$

$$g_{00}(x) = \eta_{00} + h_{00} = -1 - 2\phi(x) = -1 + \frac{2G_N M}{|x| c^2}$$

$$c^2 d\tau^2 = -g_{\mu\nu}dx^\mu dx^\nu = -g_{00}(c\,dt)^2 - g_{ii}(dx)^2$$

$$\left(\frac{d\tau}{dt}\right)^2 = -g_{00} - \left(\frac{dx}{c\,dt}\right)^2 = 1 - \frac{2G_N M}{|x|\,c^2} - \left(\frac{v}{c}\right)^2$$

$$\frac{d\tau}{dt} = \left(1 - \left(\frac{v}{c}\right)^2 - \frac{2G_N M}{|x|\,c^2}\right)^{1/2}$$

$$\simeq 1 - \left(\frac{1}{2}\frac{v^2}{c^2} - \phi(x)/c^2\right) = 1 - (T-V)/mc^2$$

$$\tau = \int_{t_1}^{t_2} d\tau = \int dt\,\frac{d\tau}{dt} \simeq \int dt\,\left(1 - \frac{1}{mc^2}(T-V)\right]$$

$$\simeq \text{const} - \frac{1}{mc^2}\int dt\,(T-V)$$

This is the Lagrangian in the non-relativistic physics

§ Gravitational red shift

O $\qquad$ S,

$\bullet$ $\qquad$ $\bullet \longrightarrow \vec{v}$

Recap the Doppler shift:

O: observer

S: source. Source moves at velocity v away from O.

Consider two time intervals dt_o (observer frame) and dt_s (source frame) of emitting a light period.

Since light comes from the source, hence. dt_s is the proper time.

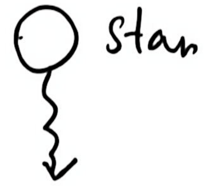
$$dt_s = \sqrt{1-\beta^2}\, dt_o$$

Suppose in S-frame, two emissions at a time interval Δt_s. Correspondingly in O-frame, the time interval $\frac{\Delta t_s}{\sqrt{1-\beta^2}}$. And the source moves away at distance $\frac{v \Delta t_s}{\sqrt{1-\beta^2}}$. Hence the time interval received by the observe O is

$$\Delta t_o = \Delta t_s / \sqrt{1-\beta^2} + \frac{v \Delta t_s}{c\sqrt{1-\beta^2}} = \Delta t_s \frac{1+\beta}{\sqrt{1-\beta^2}}$$

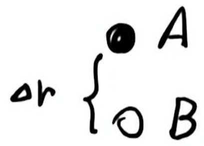
$$= \Delta t_s \sqrt{\frac{1+\beta}{1-\beta}} \quad \Rightarrow \quad \frac{\nu_o}{\nu_s} = \frac{\Delta t_s}{\Delta t_o} = \left(\frac{1-\beta}{1+\beta}\right)^{1/2}$$

- Red shift in a gravity field



photon emitted by a star.

At point



$$A: \quad E_A = h\nu_A$$

$$B \quad E_B = h\nu_B + m\Delta\phi = h\nu_B + \frac{h\nu_B}{c^2}\Delta\phi$$
$$= h\nu_A$$

$$\phi_B - \phi_A = \frac{G_N M}{c^2 r^2}\Delta r$$

$$\Rightarrow \frac{\nu_B}{\nu_A} = \frac{1}{1+\Delta\phi}$$

$$= \frac{1}{1+\frac{G_N M}{c^2 r^2}\Delta r} \simeq 1 - \frac{G_N M}{c^2 r}\frac{\Delta r}{r}$$

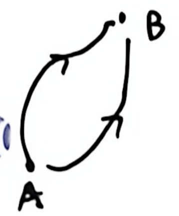
Hence, as a photon moves away from the star, the photon frequency is shifted to the red side.

- Dilation of time

$$d\tau = \left(-g_{\mu\nu}(x)\,dx^\mu dx^\nu\right)^{1/2}$$

Unlike the Minkowski space, there's no global proper time, i.e

$$(\Delta\tau)_{AB} = \int_{\tau_A}^{\tau_B} d\tau \quad \text{is path dependent}$$

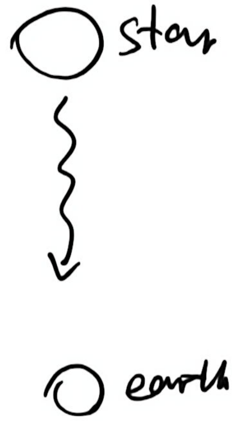


$\Delta\tau$ is coordinate invariant, but not unique

Consider a star which sends N light waves.

Observe B is on the earth, he uses a coordinate system (x, t). The metric $g_{00}(x)$ is space-dependent!

$g_{00}(\text{earth}) \neq g_{00}(\text{star})$. Then the propertime intervals of sending these light waves are



$$d\tau_{earth} = (-g_{00}(\text{earth}))^{1/2}\, dt$$

$$d\tau_{star} = (-g_{00}(\text{star}))^{1/2}\, dt$$

$$\left.\begin{array}{l} N = \nu_{star} \cdot \Delta\tau_{star} \\ N = \nu_{earth} \cdot \Delta\tau_{earth} \end{array}\right\}$$ frequency is determined via propertime

$$\Rightarrow \frac{\nu_{star}}{\nu_{earth}} = \frac{\Delta\tau_{earth}}{\Delta\tau_{star}} = \frac{(-g_{00}(\text{earth}))^{1/2}}{(-g_{00}(\text{star}))^{1/2}} = \frac{(1+2\phi_{earth})^{1/2}}{(1+2\phi_{star})^{1/2}}$$

$$= 1 + \phi_{earth} - \phi_{star}.$$

For a star $\phi_{star} = -G_N M_s / R_s$ $\qquad \phi_{earth} = -G_N \dfrac{M_{earth}}{R_{earth}}$

$$\frac{\Delta\lambda}{\lambda} = |\Delta\phi| = \frac{G_N}{c^2}\left(\frac{M_S}{R_S} - \frac{M_E}{R_E}\right)$$

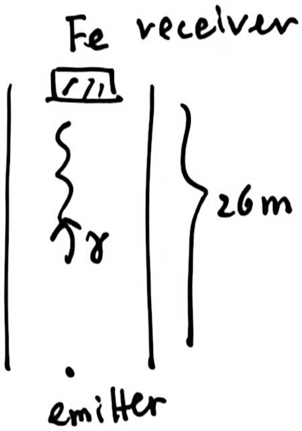
The hydrogen spectrum from white dwarf is red-shifted

E.g. 40-Eridani B $\quad \left(\dfrac{\Delta\lambda}{\lambda}\right)_{exp} \approx 7\times10^{-5}$, theory $\left(\dfrac{\Delta\lambda}{\lambda}\right) \simeq 5.6\times10^{-5}$

Test via the Mössbauer effect : ^{57}Fe recoilless γ-emission and absorption.

$$\left|\frac{\Delta \nu}{\nu}\right| = \Delta\phi/c^2 = \frac{G_N M_E}{c^2 R^2}\Delta r = \frac{g\Delta r}{c^2}$$

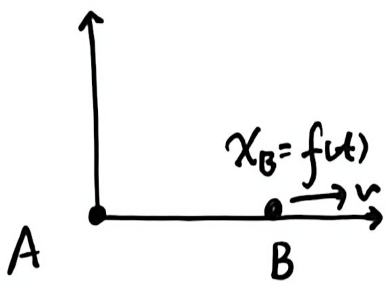
$$= \frac{10\times 26}{9\times 10^{16}} \simeq 2.5\times 10^{-15}$$



Twin paradox ~~—~~ — who is actually younger

A: stay in earth (x, t)

B: travels in space craft and then returns (X, T)



A, B coordinate transformation

$$\begin{cases} X = x - f(t) \\ T = t \end{cases}$$

This transformation looks wierd, but the metric can be non-trivial

In frame A, $-(d\tau)^2 = g_{xx}\,dx^2 + 2g_{xt}\,dx\,dt + g_{tt}\,dt^2$

In frame B $-(d\tau)^2 = g_{XX}\,dX^2 + 2g_{XT}\,dX\,dT + g_{TT}\,dT^2$

$$= g_{XX}(dx - \dot{f}\,dt)^2 + 2g_{XT}(dx - \dot{f}\,dt)\,dt + g_{TT}(dt)^2$$

$$= g_{XX}\,dx^2 + (2g_{XT} - 2g_{XX}\dot{f})\,dx\,dt + (g_{TT} - 2g_{XT}\dot{f} + g_{XX}\dot{f}^2)\,dt^2$$

$$\Rightarrow \begin{cases} g_{xx} = g_{XX} \\ g_{xt} = g_{XT} - g_{XX}\dot{f} \\ g_{tt} = g_{TT} - 2g_{XT}\dot{f} + g_{XX}\dot{f}^2 \end{cases}$$

$$\rightarrow \quad \begin{aligned} g_{XX} &= g_{xx} \\ g_{XT} &= g_{xt} + g_{xx}\dot{f} \\ g_{TT} &= g_{tt} + 2(g_{xt} + g_{xx}\dot{f})\dot{f} - g_{xx}\dot{f}^2 \\ &= g_{tt} + 2g_{xt}\dot{f} + g_{xx}\dot{f}^2 \end{aligned}$$

Assume space craft B takes off at $t_1 = T_1$, and the time return, its proper time in A's frame

$$\tau_A(B) = \int_{t_1}^{t_2} d\tau = \int_{t_1}^{t_2} dt \left[-g_{xx}\left(\frac{dx_B}{dt}\right)^2 - 2g_{xt}\left(\frac{dx_B}{dt}\right)\left(\frac{dt}{dt}\right) - g_{tt}\left(\frac{dt}{dt}\right)^2 \right]$$

$$\tau_A(B) = \int_{t_1}^{t_2} d\tau = \int_{t_1}^{t_2} dt \ \left[-\left(g_{tt} + 2 g_{xt} \dot{f} + g_{xx} \dot{f}^2\right)\right]^{1/2}$$

$$= \int_{T_1}^{T_2} dT \ \sqrt{-g_{TT}}$$

$$= \tau_B(B)$$

$$\tau_A(B) - \tau_A(A) = \int_{t_1}^{t_2} dt \ \left(\sqrt{-g_{TT}} - \sqrt{-g_{tt}}\right)$$

$$= \tau_B(B) - \tau_B(A)$$

no paradox of twins.

⊛ Equation of motion in the presence of other forces

Consider non gravitational force f^μ, we have

$$m \frac{d^2 x^\mu}{d\tau^2} - m \Gamma^\mu_{\nu\lambda} \frac{dx^\nu}{d\tau} \frac{dx^\lambda}{d\tau} = f^\mu$$

In flat space, the 4-velocity $u^\mu = \frac{dx^\mu}{d\tau} = \left(\frac{dx}{d\tau}, \frac{dt}{d\tau}\right)$

$u^\mu \cdot u_\mu = -1$ hence $f^\mu = \frac{du^\mu}{d\tau} \cdot \frac{1}{m} \Rightarrow f^\mu u_\mu = 0$

$$= g_{\mu\nu} u^\mu f^\nu = 0$$

For the general case,

$$m \, Du^\mu / D\tau = f^\mu \rightarrow m g_{\mu\nu} u^\mu \frac{Du^\nu}{D\tau} = g_{\mu\nu} u^\mu f^\nu$$

$$\frac{m}{2}\frac{D}{D\tau}(g_{\mu\nu}u^\mu u^\nu) = g_{\mu\nu}u^\mu f^\nu$$

scalar $\frac{D}{D\tau} = \frac{\partial}{\partial\tau} \Rightarrow \frac{\partial}{\partial\tau}(g_{\mu\nu}u^\mu u^\nu) = 0$

· EM theory in flat space

$$\partial_\mu F^{\mu\nu} = J^\nu, \qquad \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0.$$

In curved space.

$$D_\mu F^{\mu\nu} = J^\nu$$

$$F^{\mu\nu} = D^\mu A^\nu - D^\nu A^\mu \neq \partial^\mu A^\nu - \partial^\nu A^\mu$$

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu \\ &= \partial_\mu A_\nu - \Gamma^\lambda_{\mu\nu} A_\lambda \\ &\quad - (\partial_\nu A_\mu - \Gamma^\lambda_{\nu\mu} A_\lambda) \end{aligned}$$

$$\begin{aligned} &D_\mu F^{\mu\nu} \\ &= \partial_\mu F^{\mu\nu} + \Gamma^\mu_{\mu\lambda} F^{\lambda\nu} \\ &= \partial_\mu F^{\mu\nu} + \left(\frac{1}{\sqrt{-g}}\partial_\lambda \sqrt{-g}\right) F^{\lambda\nu} \\ &= \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\, F^{\mu\nu}) \end{aligned}$$

$$\Rightarrow \boxed{\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\, F^{\mu\nu}) = J^\nu}$$

$$\partial_\nu\partial_\mu(\sqrt{-g}\, F^{\mu\nu}) = \partial_\nu(\sqrt{-g}\, J^\nu) = 0$$

$$D_\mu F_{\nu\lambda} + D_\nu F_{\lambda\mu} + D_\lambda F_{\mu\nu} = 0$$

$$\Rightarrow \boxed{\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0}$$

$$\sqrt{0} \quad D_\mu J^\mu = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\, J^\mu) = 0$$

Define global charge

$$Q(t) = \int d^3x \sqrt{-g}\, J^0$$

$$dQ/dt = \int d^3x\, \partial_0(\sqrt{-g}\,J^0) = -\int d^3x\, \partial_i(\sqrt{-g}\,J^i) = 0$$

§ Energy-momentum tensor

Consider a system of particles.

$$T^{\mu 0}_{\text{matter}} = \sum_n P_n^\mu\, \delta^{(3)}(\vec{x} - \vec{x}_n(t))$$

$$= \int d\tau \sum_n P_n^\mu \frac{dx_n^{(0)}(\tau)}{d\tau} \underbrace{\delta^{(3)}(\vec{x} - \vec{x}_n(t))\,\delta(t - x_n^{(0)}(\tau))}_{\delta^{(4)}(x - x_n(\tau))}$$

$$T^{\mu i}_{\text{matter}} = \sum_n P_n^\mu\, dx_n^i(t)/dt\; \delta^{(3)}(\vec{x} - \vec{x}_n(t)) \cancel{= \int d\tau \sum_n P_n^\mu} d\,\text{\textbackslash}$$

$$= \int d\tau \sum_n P_n^\mu \frac{dx_n^i(\tau)}{d\tau}\, \delta^{(4)}(x - x_n(\tau)) \qquad \frac{dx_n^i}{dt}\cdot\frac{dt}{d\tau}$$

$$\Rightarrow T^{\mu\nu}_{\text{matter}} = \int d\tau \sum_n P_n^\mu \frac{dx_n^\nu(\tau)}{d\tau}\, \delta^{(4)}(x - x_n(\tau))$$

$$= m\int d\tau \sum_n \frac{dx_n^\mu(\tau)}{d\tau}\frac{dx_n^\nu(\tau)}{d\tau}\, \delta^{(4)}(x - x_n(\tau))$$

$$= T^{\nu\mu}_{\text{matter}}$$

In flat space

$$\partial_\mu T^{\mu\nu}_{\text{matter}} = \frac{\partial}{\partial x^\mu}\, m\int d\tau \sum_n \frac{dx_n^\mu}{d\tau}\frac{dx_n^\nu}{d\tau}\, \delta^{(4)}(x - x_n(\tau))$$

$$\frac{\partial}{\partial x^\mu}\delta^{(4)}(x - x_n(\tau)) = -\frac{\partial}{\partial x_n^\mu}\delta^{(4)}(x - x_n(\tau))$$

$$\partial_\mu T^{\mu\nu}_{matter} = m\int d\tau \sum_n \frac{dx_n^\mu}{d\tau}\frac{dx_n^\nu}{d\tau}\left(-\frac{\partial}{\partial x_n^\mu(\tau)}\delta^{(4)}(x-x_n(\tau))\right)$$

$$= m\int d\tau \sum_n \frac{dx_n^\nu}{d\tau}\left(-\frac{d}{d\tau}\delta^{(4)}(x-x_n(\tau))\right)$$

$$= \int d\tau \sum_n \frac{dp_n^\nu}{d\tau}\delta^{(4)}(x-x_n(\tau))$$

$$= \sum_n \int d\tau \; f_n^\nu \,\delta^{(4)}(x-x_n(\tau))$$

$$P^\mu_{matter}(t) = \int d^3x \; T^{0\mu}_{matter} = \int d^3x \; T^{\mu 0}_{matter}$$

$$= \int d^3x \int d\tau \sum_n p_n^\mu \frac{dx_n^0}{d\tau}\delta^{(4)}(x-x_n(\tau))$$

$$= \int d^4x \sum_n p_n^\mu \,\delta^{(4)}(x-x_n(\tau)) = \sum_n p_n^\mu$$

In the presence of EM field.

$$f^\nu = q\, F^\nu{}_\lambda \frac{dx^\lambda}{d\tau}$$

$$\Rightarrow \partial_\mu T^{\mu\nu}_{matter} = \int d\tau \sum_n q_n F^\nu{}_\lambda \frac{dx_n^\lambda}{d\tau}\delta^{(4)}(x-x_n(\tau))$$

$$= F^\nu{}_\lambda \underbrace{\int d\tau \sum_n q_n \frac{dx_n^\lambda}{d\tau}\delta^{(4)}(x-x_n(\tau))}_{J^\lambda} \Rightarrow \boxed{\partial_\mu T^{\mu\nu}_{matter} = F^\nu{}_\lambda J^\lambda}$$

Define EM field momentum - energy tensor

$$T^{\mu\nu}_{(em)} = F^{\mu}{}_{\lambda} F^{\nu\lambda} - \frac{1}{4}\eta^{\mu\nu} F_{\lambda\sigma} F^{\lambda\sigma}$$

GR convention $x^{\mu} = (x, ct)$ $\partial^{\mu} = (\partial_x, -\frac{\partial_t}{c})$ 和场论

$A^{\mu} = (A, \varphi)$ $\partial_{\mu} = (\partial_x, \frac{1}{c}\partial_t)$ 习惯不同

$$F^{\mu\nu} = \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}, \qquad F^{0i} = \partial^0 A^i - \partial^i A^0 = -\frac{1}{c}\partial_t \vec{A} - \nabla\varphi$$

$$= \vec{E}$$

$$F^{ij} = \partial^i A^j - \partial^j A^i = \varepsilon_{ijk} B^k$$

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ & 0 & B_z & -B_y \\ & & 0 & B_x \\ & & & 0 \end{pmatrix} \qquad F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ & 0 & B_z & -B_y \\ & & 0 & B_x \\ & & & 0 \end{pmatrix}$$

$$\tilde{F}^{\mu\nu} = \varepsilon^{\mu\nu\lambda\sigma} F_{\lambda\sigma} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ & 0 & -E_z & E_y \\ & & 0 & -E_x \\ & & & 0 \end{pmatrix}$$

$$\frac{1}{4} F^{\mu\nu} F_{\mu\nu} = \frac{1}{4}(-\vec{E}^2 + \vec{B}^2) \times 2 = -\frac{1}{2}(\vec{E}^2 - \vec{B}^2)$$

$$\eta^{00} = -1$$

$$T^{00}_{(em)} = F^0{}_i F^{0i} + \ldots - \frac{1}{2}(E^2 - B^2)$$

$$= \eta^{00} F_{0i} F^{0i} - \frac{1}{2}(E^2 - B^2)$$

$$= -(-E^2) - \frac{1}{2}(E^2 - B^2) = +\frac{1}{2}(E^2 + B^2)$$

$$T^{0i} = F^0{}_j F^{ji} - \eta^{0i} \cancel{(-\tfrac{1}{2})} (E^2 - B^2)$$

$$T^{0i} = \eta^{00} F_{0j} F^{ij} = -(-)E_j \, \underset{ij}{\epsilon_{jik}} B_k$$

$$= \epsilon_{ijk} E_j B_k = (\vec{E} \times \vec{B})_i$$

$$\partial_\mu T^{\mu\nu}_{em} = \partial_\mu F^\mu{}_\lambda F^{\nu\lambda} + F^\mu{}_\lambda \partial_\mu F^{\nu\lambda} - 1/2 \, \eta^{\mu\nu} (\partial_\mu F^{\lambda\rho}) F_{\lambda\rho}$$

$$= \partial_\mu F^\mu{}_\lambda F^{\nu\lambda} + F_{\mu\lambda} \partial^\mu F^{\nu\lambda} - 1/2 (\partial^\nu F^{\lambda\rho}) F_{\lambda\rho}$$

$$\swarrow$$

$$-F_{\lambda\rho} \partial^\rho F^{\nu\lambda} = -1/2 \, F_{\lambda\rho} (\partial^\rho F^{\nu\lambda} + \partial^\lambda F^{\rho\nu})$$

$$= \partial_\mu F^\mu{}_\lambda F^{\nu\lambda} - 1/2 \, F_{\lambda\rho} (\partial^\rho F^{\nu\lambda} + \partial^\lambda F^{\rho\nu} + \partial^\nu F^{\lambda\rho})$$

$\rightarrow$ Bianchi identity

$$= -J_\lambda F^{\nu\lambda}$$

$$\partial_\mu F^{\mu\nu} = -J^\nu$$

$$\eta_{\nu\lambda} \partial_\mu F^{\mu\nu} = \partial_\mu F^\mu{}_\lambda = -J_\lambda$$

(度规 (1 -1 -1 -1)

$$\partial_\mu F^{\mu\nu} = J^\nu$$

度规 (-1, 1, 1, 1)

$$\partial_\mu F^{\mu\nu} = -J^\nu \,)$$

$$\Rightarrow \boxed{\begin{aligned} &\partial_\mu (T^{\mu\nu}_{matter} + T^{\mu\nu}_{em}) \\ &= F^\nu{}_\lambda J^\lambda - J_\lambda F^{\nu\lambda} = 0 \end{aligned}}$$

$$T^{tot,\,\mu\nu} = \int d\tau \sum_n P_n^\mu \frac{dx_n^\nu}{d\tau} \delta^{(4)}(x - x_n(\tau))$$

$$+ F^\mu{}_\lambda F^{\nu\lambda} - 1/4 \, \eta^{\mu\nu} F_{\lambda\sigma} F^{\lambda\sigma}$$

For curved space, we expect

$$D_\mu T^{\mu\nu}_{tot} = 0$$

$$\partial_\mu T^{\mu\nu}_{tot} + \Gamma^\mu_{\mu\rho} T^{\rho\nu}_{tot} + \Gamma^\nu_{\mu\rho} T^{\mu\rho}_{tot} = 0$$

$$\frac{1}{\sqrt{-g}} \partial_\rho (\sqrt{-g})\, T^{\rho\nu}_{tot}$$

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g}\, T^{\mu\nu}_{tot}) = - \Gamma^\nu_{\mu\rho} T^{\mu\rho}_{tot}$$

not conserved

we have to include the momentum, energy of the gravity field. but now we don't know how to construct it.

⊛ Generalization of a scalar field Eq

$$(\partial_\mu \partial^\mu \pm m^2)\, \phi(x) = 0$$

$$D_\mu \partial^\mu \phi(x) + m^2 \phi(x) = 0$$

$$\boxed{\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g}\, \partial^\mu \phi(x)) + m^2 \phi(x) = 0}$$