# Lecture 8 Curvature tensor and Einsteino Equation

## §. Geodesic deviation

## § Curvature tensor and symmeties

$$R_{\mu\nu\lambda\sigma} = -R_{\nu\mu\lambda\sigma} = -R_{\mu\nu\sigma\lambda} = R_{\lambda\sigma\mu\nu}$$

$$R_{\mu\nu\lambda\sigma} + R_{\mu\lambda\sigma\nu} + R_{\mu\sigma\nu\lambda} = 0$$

## § Ricci tensor

$$\boxed{R_{\mu\nu} = g^{\lambda\sigma} R_{\mu\lambda\nu\sigma}, \quad R = g^{\mu\nu} R_{\mu\nu}}$$

## § Construction of Einstein Equation

$$D_\mu G^{\mu\nu} = 0$$

$$G^{\mu\nu} = \boxed{R^{\mu\nu} - \tfrac{1}{2} g^{\mu\nu} R = 8\pi G_N T^{\mu\nu}}$$

§ Geodesic deviation

Consider a group of geodesics $x^\mu(\tau, h)$ where $\tau$ is the proper time along a geodesic, and $h$ distinguishes different geodesics.

$$\frac{\partial^2 x^\mu(\tau,h)}{\partial^2 \tau^2} + \Gamma^\mu_{\nu\lambda}\frac{\partial x^\nu}{\partial \tau}\frac{\partial x^\lambda}{\partial \tau} = 0$$

Define tangent vector $u^\mu(\tau,h) = \partial x^\mu(\tau,h)/\partial\tau$, then the geodesic becomes $\boxed{\partial u^\mu(\tau,h)/\partial\tau + \Gamma^\mu_{\nu\lambda} u^\nu u^\lambda = 0.}$

Define $v^\mu(\tau,h) = \partial x^\mu(\tau,h)/\partial h$, $\Rightarrow$

$$\frac{\partial v^\mu}{\partial\tau} = \frac{\partial^2 x^\mu}{\partial\tau\partial h} = \frac{\partial u^\mu}{\partial h} \Rightarrow \frac{Dv^\mu}{D\tau} = \frac{\partial v^\mu}{\partial\tau} + \Gamma^\mu_{\nu\lambda}\frac{\partial x^\nu}{\partial\tau} v^\lambda$$

$$= \frac{\partial u^\mu}{\partial h} + \Gamma^\mu_{\nu\lambda} u^\nu v^\lambda$$

$$\frac{D^2 v^\mu}{D\tau^2} = \frac{\partial}{\partial\tau}\left(\frac{Dv^\mu}{\partial\tau}\right) + \Gamma^\mu_{\rho\sigma}\frac{\partial x^\rho}{\partial\tau}\frac{Dv^\sigma}{D\tau}$$

$$= \frac{\partial}{\partial\tau}\left(\frac{\partial u^\mu}{\partial h} + \Gamma^\mu_{\nu\lambda}u^\nu v^\lambda\right) + \Gamma^\mu_{\rho\sigma}\frac{\partial x^\rho}{\partial\tau}\left(\frac{\partial u^\sigma}{\partial h} + \Gamma^\sigma_{\nu\lambda}u^\nu v\right.$$

$$= \frac{\partial}{\partial h}\frac{\partial u^\mu}{\partial\tau} + \frac{\partial}{\partial x^\sigma}\Gamma^\mu_{\nu\lambda}u^\sigma u^\nu v^\lambda + \Gamma^\mu_{\nu\lambda}\frac{\partial u^\nu}{\partial\tau}v^\lambda + \Gamma^\mu_{\nu\lambda}u^\nu.$$

$$+ \Gamma^\mu_{\rho\sigma}\frac{\partial x^\rho}{\partial\tau}\frac{\partial u^\sigma}{\partial h} + \Gamma^\mu_{\rho\sigma}\Gamma^\sigma_{\nu\lambda}u^\rho u^\nu v^\lambda$$

$$\rightarrow \frac{\partial}{\partial\tau}u^\mu = -\Gamma^\mu_{\nu\lambda}u^\nu u^\lambda$$

$$\cancel{\frac{\partial}{\partial h}\left(\frac{\partial u^\mu}{\partial\tau}\right) = -\frac{\partial}{\partial h}\left(\Gamma^\mu_{\nu\lambda}u^\nu u^\lambda\right)} = -\cancel{\left(\frac{\partial}{\partial x^\sigma}\Gamma^\mu_{\nu\lambda}v^\sigma u^\nu u^\lambda\right.}$$

$$+ \Gamma^\mu_{\nu\lambda}\cancel{\frac{\partial u^\nu}{\partial h}u^\lambda} + \Gamma^\mu_{\nu}$$

$$\frac{D^2 v^\mu}{D\tau^2} = -\frac{\partial}{\partial h}\left(\Gamma^\mu_{\nu\lambda} u^\nu u^\lambda\right) + \partial_\sigma \Gamma^\mu_{\nu\lambda} u^\sigma u^\nu v^\lambda$$

$$+ \Gamma^\mu_{\nu\lambda}\left(-\Gamma^\nu_{\rho\sigma} u^\rho u^\sigma\right) v^\lambda + \underbrace{2\Gamma^\mu_{\nu\lambda} u^\nu \frac{\partial u^\lambda}{\partial h}}$$

$$+ \Gamma^\mu_{\rho\sigma} \Gamma^\sigma_{\nu\lambda} u^\rho u^\nu v^\lambda$$

$$\frac{\partial}{\partial h}\left(\Gamma^\mu_{\nu\lambda} u^\nu u^\lambda\right) = \partial_\sigma \Gamma^\mu_{\nu\lambda} v^\sigma u^\nu u^\lambda + \underbrace{\Gamma^\mu_{\nu\lambda} \frac{\partial}{\partial h} u^\nu u^\lambda + \Gamma^\mu_{\nu\lambda} u^\nu \frac{\partial u^\lambda}{\partial h}}$$

$$\cancel{= \partial_\sigma \Gamma^\mu_{\nu\lambda} v^\sigma u^\nu u^\lambda + \Gamma^\mu_{\nu\lambda} v}$$

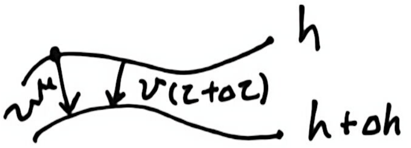
cancel.

$$\Rightarrow \frac{D^2 v^\mu}{D\tau^2} = \left(\partial_\rho \Gamma^\mu_{\nu\lambda} - \partial_\lambda \Gamma^\mu_{\nu\rho}\right) u^\rho u^\nu v^\lambda$$

$$+ \left(\Gamma^\mu_{\rho\sigma} \Gamma^\sigma_{\nu\lambda} - \Gamma^\mu_{\sigma\lambda} \Gamma^\sigma_{\rho\nu}\right) u^\rho u^\nu v^\lambda$$

$$= R^\mu{}_{\nu\rho\lambda} u^\nu u^\rho v^\lambda$$

2nd order derivative of $v^\mu$ is proportional to the Riemann curvature tensor.




The seperation between the geodecics changes.

$v^\mu$ can be viewed as the relative coordinate between two close geodesic $x^\mu(\tau, h+\delta h) - x^\mu(\tau, h) = v^\mu$.

# Properties of the curvature tensor

- $[D_\mu\ D_\nu]\, A^\rho = R^\rho{}_{\sigma\mu\nu} A^\sigma$ , hence $\boxed{R^\rho{}_{\sigma\mu\nu} = -R^\rho{}_{\sigma\nu\mu}}$

- Consider $\phi = \xi_\mu \xi^\mu = g_{\mu\nu} \xi^\mu \xi^\nu$

$$D_\lambda D_\rho \phi = D_\lambda \left(2 g_{\mu\nu} (D_\rho \xi^\mu) \xi^\nu\right) = 2 g_{\mu\nu} (D_\lambda D_\rho \xi^\mu) \xi^\nu + 2 g_{\mu\nu} D_\rho \xi^\mu D_\lambda \xi^\nu$$

$$\Rightarrow [D_\lambda, D_\rho]\, \phi = 2 g_{\mu\nu} \{[D_\lambda, D_\rho]\, \xi^\mu\} \xi^\nu$$

$$= 2 g_{\mu\nu} \xi^\nu R^\mu{}_{\sigma\lambda\rho} \xi^\sigma = 2 \xi^\nu \xi^\sigma R_{\nu\sigma\lambda\rho}$$

Since $\phi$ is a scalar, $[D_\lambda, D_\rho]\, \phi = 0 \Rightarrow \xi^\nu \xi^\sigma R_{\nu\sigma\lambda\rho} = 0$

$$\Rightarrow \boxed{R_{\nu\sigma\lambda\rho} = -R_{\sigma\nu\lambda\rho}}$$

- We define

$$\xi_{[\mu;\nu;\lambda]} = [\xi_{\mu;\nu;\lambda} - \xi_{\mu;\lambda;\nu}] + [\xi_{\nu;\lambda;\mu} - \xi_{\nu;\mu;\lambda}] + [\xi_{\lambda;\mu;\nu} - \xi_{\lambda;\nu;\mu}]$$

$$= [D_\lambda, D_\nu]\, \xi_\mu + [D_\mu, D_\lambda]\, \xi_\nu + [D_\nu, D_\mu]\, \xi_\lambda$$

$$= -R_\mu{}^\sigma{}_{\lambda\nu} \xi_\sigma - R^\sigma{}_{\nu\mu\lambda} \xi_\sigma - R^\sigma{}_{\lambda\nu\mu} \xi_\sigma$$

$$A_{[\mu\nu\lambda]} = D_\lambda A_{\mu\nu} + D_\mu A_{\nu\lambda} + D_\nu A_{\lambda\mu}$$

$$A_{\mu\nu} = D_\mu A_\nu - D_\nu A_\mu = \partial_\mu A_\nu - \Gamma^\lambda_{\mu\nu} A_\lambda - (\partial_\nu A_\mu - \Gamma^\lambda_{\nu\mu} A_\lambda)$$

$$= \partial_\mu A_\nu - \partial_\nu A_\mu \Rightarrow A_{\mu\nu} = -A_{\nu\mu}$$

$\Rightarrow$ ~~$[\, A_{[\mu\nu\lambda]} = D_\lambda$~~

$\lambda \to \mu,\ \mu \to \nu,\ \nu \to \lambda$

$$D_\lambda A_{\mu\nu} = \partial_\lambda A_{\mu\nu} - \Gamma^\sigma_{\lambda\mu} A_{\sigma\nu} - \Gamma^\sigma_{\lambda\nu} A_{\mu\sigma}$$

$$D_\mu A_{\nu\lambda} = \partial_\mu A_{\nu\lambda} - \Gamma^\sigma_{\mu\nu} A_{\sigma\lambda} - \Gamma^\sigma_{\mu\lambda} A_{\nu\sigma}$$

$$D_\nu A_{\lambda\mu} = \cancel{\partial_\lambda A_{\mu\nu}}\ \partial_\nu A_{\lambda\mu} - \Gamma^\sigma_{\nu\lambda} A_{\sigma\mu} - \Gamma^\sigma_{\nu\mu} A_{\lambda\sigma}$$

if $A_{\mu\nu} = -A_{\nu\mu} \Rightarrow D_\lambda A_{\mu\nu} + D_\mu A_{\nu\lambda} + D_\nu A_{\lambda\mu}$

$$= \partial_\lambda A_{\mu\nu} + \partial_\mu A_{\nu\lambda} + \partial_\nu A_{\lambda\mu}$$

then $A_{[\mu\nu\lambda]} = \partial_\lambda [\partial_\mu A_\nu - \partial_\nu A_\mu] + \partial_\mu [\partial_\nu A_\lambda - \partial_\lambda A_\nu] + \partial_\nu [\partial_\lambda A_\mu - \partial_\mu A_\lambda]$

$$= 0$$

Then $\xi_{[\mu;\nu;\lambda]} = D_\lambda D_\nu \xi_\mu - D_\nu D_\lambda \xi_\mu + D_\mu D_\lambda \xi_\nu - D_\lambda D_\mu \xi_\nu + D_\nu D_\mu \xi_\lambda - D_\mu D_\nu \xi_\lambda$

$$= D_\lambda (D_\nu \xi_\mu - D_\mu \xi_\nu) + D_\mu (D_\lambda \xi_\nu - D_\nu \xi_\lambda) + D_\nu (D_\mu \xi_\lambda - D_\lambda \xi_\mu)$$

$$= \cancel{\partial_\lambda}\, D_\lambda (\partial_\nu \xi_\mu - \partial_\mu \xi_\nu) + D_{\cancel{\lambda}\mu} (\partial_\lambda \xi_\nu - \partial_\nu \xi_\lambda) + D_\nu (\partial_\mu \xi_\lambda - \partial_\lambda \xi_\mu)$$

$$= \partial_\lambda (\partial_\nu \xi_\mu - \partial_\mu \xi_\nu) + \partial_\mu (\partial_\lambda \xi_\nu - \partial_\nu \xi_\lambda) + \partial_\nu (\partial_\mu \xi_\lambda - \partial_\lambda \xi_\mu)$$

$$= 0$$

$$\Rightarrow \quad R^{\sigma}{}_{\mu\lambda\nu} + R^{\sigma}{}_{\lambda\nu\mu} + R^{\sigma}{}_{\nu\mu\lambda} = 0$$

or $\quad R_{\sigma\mu\lambda\nu} + R_{\sigma\lambda\nu\mu} + R_{\sigma\nu\mu\lambda} = 0$

and $\quad R_{\mu\sigma\lambda\nu} + R_{\lambda\sigma\nu\mu} + R_{\nu\sigma\mu\lambda} = 0$ .

- Check $\quad R_{\mu\nu\lambda\rho} - R_{\lambda\rho\mu\nu} = \ ?$

$$R_{\mu\nu\lambda\rho} - R_{\lambda\rho\mu\nu} = -(R_{\mu\lambda\rho\nu} + R_{\mu\rho\nu\lambda}) + \cancel{(R_{\lambda\mu\nu\rho} + R_{\lambda\nu\rho\mu})} \ (R_{\mu\rho\nu\lambda} + R_{\nu\rho\lambda\mu})$$

$$= -R_{\mu\lambda\rho\nu} + R_{\nu\rho\lambda\mu}$$

$$= (R_{\rho\lambda\nu\mu} + R_{\nu\lambda\mu\rho}) - (R_{\nu\lambda\mu\rho} + R_{\nu\mu\rho\lambda})$$

$$= R_{\rho\lambda\nu\mu} - R_{\nu\mu\rho\lambda} = R_{\lambda\rho\mu\nu} - R_{\mu\nu\lambda\rho}$$

$\Rightarrow \quad R_{\mu\nu\lambda\rho} - R_{\lambda\rho\mu\nu} = 0 \quad$ or $\quad R_{\mu\nu\lambda\rho} = R_{\lambda\rho\mu\nu}$ .

- Summary,

$$\begin{cases} R_{\mu\nu\lambda\rho} = -R_{\mu\nu\rho\lambda} = -R_{\nu\mu\lambda\rho} = R_{\lambda\rho\mu\nu} \\ R_{\mu\nu\lambda\rho} + R_{\mu\lambda\rho\nu} + R_{\mu\rho\nu\lambda} = 0 \end{cases}$$

$\mu\nu$ — antisymmetric 6

$\rho\lambda$ — antisymmetric 6 $\quad \Big\} \quad$ ~~6×6 = 36~~

$\mu\nu ; \rho\lambda$ symmetrized $\quad \Big\} \quad H_6^2 = \dfrac{6\times 7}{2} = 21$

Bianchi identity $\rightarrow \binom{n}{4}$ For $n=4$, $=1 \Rightarrow 21-1=20$

For the general case, # of degrees of freedom $\dfrac{n^2(n^2-1)}{12}$

Contraction: → Ricci tensor

$$R_{\mu\lambda} = g^{\nu\rho} R_{\mu\nu\lambda\rho} = R_{\mu\nu\lambda}{}^{\nu} = R_{\mu}{}^{\nu}{}_{\lambda\nu}$$

$$R_{\mu\lambda} = g^{\nu\rho} R_{\nu\mu\rho\lambda} = R^{\rho}{}_{\mu\rho\lambda} \cancel{= R^{\rho}}$$

$$= R^{\nu}{}_{\mu\nu\lambda} = -R^{\nu}{}_{\mu\lambda\nu} \cancel{= R^{\nu}_{\mu}}$$

Ricci scalar curvature $R = g^{\mu\lambda} R_{\mu\lambda}$

$R_{\mu\nu\lambda\rho}$ → The only possible contraction to a scalar

Now we define a 2nd rank symmetric tensor from the Ricci tensor

$$\boxed{G^{\mu\nu} = R^{\mu\nu} - \tfrac{1}{2} g^{\mu\nu} R}$$

Why define $G^{\mu\nu}$ in this form? $D_\mu G^{\mu\nu} = 0$, which means that it is conserved!

Let's check: $[D_\lambda D_\rho]\, \xi^{\mu} = R^{\mu}{}_{\sigma\lambda\rho}\, \xi^{\sigma}$

$$\rightarrow [D_\lambda, D_\rho]\, \xi_\mu = R_{\mu\nu\lambda\rho}\, \xi^{\nu}$$

$$D_\sigma([D_\lambda, D_\rho]\, \xi_\mu) = D_\sigma(R_{\mu\nu\lambda\rho}\, \xi^{\nu})$$

$$= D_\sigma(R_{\mu\nu\lambda\rho})\, \xi^{\nu} + R_{\mu\nu\lambda\rho}\, D_\sigma \xi^{\nu} \qquad (*)$$

On the other hand

$$[D_\lambda, D_\rho] D_\sigma \xi_\mu = R_{\sigma\nu\lambda\rho} D^\nu \xi_\mu + R_{\mu\nu\lambda\rho} D_\sigma \xi^\nu \quad (**)$$

~~Anti-sym~~ (*) → ($\sigma\lambda\rho$) cyclic permutation

$$D_\sigma([D_\lambda, D_\rho]\xi_\mu) + D_\lambda([D_\rho, D_\sigma]\xi_\mu) + (D_\rho[D_\sigma, D_\lambda]\xi_\mu)$$

$$\begin{aligned} &= D_\sigma(R_{\mu\nu\lambda\rho})\xi^\nu + R_{\mu\nu\lambda\rho} D_\sigma \xi^\nu \\ &+ D_\lambda(R_{\mu\nu\rho\sigma})\xi^\nu + R_{\mu\nu\rho\sigma} D_\lambda \xi^\nu \\ &+ D_\rho(R_{\mu\nu\sigma\lambda})\xi^\nu + R_{\mu\nu\sigma\lambda} D_\rho \xi^\nu \end{aligned}$$

$\sigma \to \lambda$
$\lambda \to \rho$
$\rho \to \sigma$

(**) → ($\sigma\lambda\rho$) cyclic

$$[D_\lambda, D_\rho] D_\sigma \xi_\mu + [D_\rho, D_\sigma] D_\lambda \xi_\mu + [D_\sigma D_\lambda] D_\rho \xi_\mu$$

$$\begin{aligned} = \cancel{R_{\lambda\nu\rho\sigma} D^\nu \xi}\ & R_{\sigma\nu\lambda\rho} D^\nu \xi_\mu + R_{\mu\nu\lambda\rho} D_\sigma \xi^\nu \\ + \ & R_{\lambda\nu\rho\sigma} D^\nu \xi_\mu + R_{\mu\nu\rho\sigma} D_\lambda \xi^\nu \\ + \ & R_{\rho\nu\sigma\lambda} D^\nu \xi_\mu + R_{\mu\nu\sigma\lambda} D_\rho \xi^\nu \ \cancel{= 0} \end{aligned}$$

clyclic relatin $R_{\sigma\nu\lambda\rho} + R_{\lambda\nu\rho\sigma} + R_{\rho\nu\sigma\lambda} = 0$

$$\cancel{R_{\mu\nu\lambda\rho} + R_{\mu\nu\rho\sigma} + R_{\mu\nu\sigma\lambda} = 0}$$

⇒ ~~on the other hand~~

$$\cancel{[D_\lambda D_\rho] D_\sigma + [D_\rho D_\sigma] D_\lambda + [D_\sigma D_\lambda] D_\rho}$$

$$= \cancel{D_\sigma [D_\lambda D_\rho] + D_\lambda [D_\rho D_\sigma] + D_\rho [D_\sigma, D_\lambda]}$$

$$[D_\lambda, D_\rho] D_\sigma \xi_\mu + [D_\rho\, D_\sigma] D_\lambda \xi_\mu + [D_\sigma\, D_\lambda] D_\rho \xi_\mu$$

$$= R_{\mu\nu\lambda\rho} D_\sigma \xi^\nu + R_{\mu\nu\rho\sigma} D_\lambda \xi^\nu + R_{\mu\nu\sigma\lambda} D_\rho \xi^\nu$$

Furthermore $[D_\lambda\, D_\rho] D_\sigma + [D_\rho\, D_\sigma] D_\lambda + [D_\sigma\, D_\lambda] D_\rho$

$$= D_\sigma [D_\lambda\, D_\rho] + D_\lambda [D_\rho\, D_\sigma] + D_\rho [D_\sigma\, D_\lambda]$$

$$\Rightarrow \left[ D_\sigma R_{\mu\nu\lambda\rho} + D_\lambda R_{\mu\nu\rho\sigma} + D_\rho R_{\mu\nu\sigma\lambda} \right] \xi^\nu = 0$$

i.e. $D_\sigma R_{\mu\nu\lambda\rho} + D_\lambda R_{\mu\nu\rho\sigma} + D_\rho R_{\mu\nu\sigma\lambda} = 0$

$$g^{\nu\rho} [ D_\sigma R_{\mu\nu\lambda\rho} + D_\lambda R_{\mu\nu\rho\sigma} + D_\rho R_{\mu\nu\sigma\lambda} ] = 0$$

$$D_\sigma R_{\mu\lambda} \mp D_\lambda R_{\mu\sigma} - D_\rho R^\rho{}_{\mu\sigma\lambda} = 0$$

$$g^{\mu\lambda} ( D_\sigma R_{\mu\lambda} - D_\lambda R_{\mu\sigma} - D_\rho R^\rho{}_{\mu\sigma\lambda} ) = 0$$

$$D_\sigma R - D_\lambda R^\lambda{}_\sigma - D_\rho R^\rho{}_\sigma = 0$$

$$\frac{1}{2} D_\sigma R - D_\mu R^\mu{}_\sigma = 0$$

$$D_\mu R^\mu{}_\nu - \tfrac{1}{2} D_\nu R = 0$$

$$\Rightarrow D_\mu R^{\mu\nu} - \tfrac{1}{2} D^\nu R = 0$$

$$D_\mu ( R^{\mu\nu} - \tfrac{1}{2} g^{\mu\nu} R ) = 0 \Rightarrow D_\mu G^{\mu\nu} = 0$$

⊛ 2D curved space

$$R_{1212} = R_{2121} = -R_{1221} = -R_{2112}$$

$$g = g_{11} g_{22} - g_{12} g_{21} = \det g_{\mu\nu}$$

After complicated ~~by~~ but straightforward calculation, we have

$R_{1212} = gK$, where $K$ is the Gaussian curvature.

Then $R_{\mu\nu\lambda\rho} = K(g_{\mu\lambda} g_{\nu\rho} - g_{\mu\rho} g_{\nu\lambda})$

Ricci tensor $R_{\mu\nu} = g^{\sigma\rho} R_{\mu\sigma\nu\rho} = g^{\sigma\rho} K(g_{\mu\nu} g_{\sigma\rho} - g_{\mu\rho} g_{\sigma\nu})$

$$= K(2g_{\mu\nu} - g_{\mu\nu}) = K g_{\mu\nu}$$

$$R = g^{\mu\nu} R_{\mu\nu} = 2K$$

$$\frac{D^2 v^\mu}{D\tau^2} = R^\mu{}_{\nu\rho\lambda}\, u^\nu u^\rho v^\lambda$$

$$u^\mu = \frac{\partial x^\mu}{\partial \tau} \qquad v^\mu = \partial x^\mu / \partial h$$

$$D^2 \delta x^\mu / D\tau^2 = R^\mu_{\nu\rho\lambda} \frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau}\,\delta x^\lambda = K^\mu{}_\lambda\, \delta x^\lambda$$

$$d^2 x^i / dt^2 = -\partial^i \phi \big|_x \rightarrow \boxed{\nabla^2 \phi(x) = 4\pi G_N\, \rho(x)}$$

$$\delta x^i = \tilde{x}^i - x^i$$

$$d^2 \tilde{x}^i / dt^2 = -\partial^i \phi \big|_{\tilde{x}}$$

$$d^2(\tilde{x}^i - x^i)/dt^2 \simeq -\delta x^j\, \partial^i \partial_j\, \phi(x) \big|_x$$

$$\Rightarrow \quad K^i{}_j = -\partial^i \partial_j \phi$$

In empty $\quad \nabla^2 \phi = -\partial^i \partial_i \phi = K^i{}_i = 0$

$$K^\mu{}_\mu = R^\mu{}_{\nu\rho\mu} \frac{dx^\nu}{d\tau}\,\frac{dx^\rho}{d\tau} = 0$$

$$\rightarrow -R_{\nu\rho} \frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0 \Rightarrow \boxed{R_{\nu\rho} = 0} \rightarrow$$ in a space free of matte

$\Rightarrow$ gravitional field equatin free of matter

$$\boxed{R_{\mu\nu} = 0} \rightarrow$$ Curved space free of matter

weak field limit $g_{00}(x) = -(1+2\phi(x))$

$\Rightarrow \nabla^2\phi = 4\pi G_N \rho \Rightarrow \nabla^2 g_{00} = 8\pi G_N T_{00} \rightarrow$ energy-momentum tensor

Conjecture
$\rightarrow \quad R^{\mu\nu} = \alpha T^{\mu\nu}$

Conservation law $\quad D_\mu T^{\mu\nu} = 0$, but $D_\mu R^{\mu\nu} \neq 0$

Now $\quad G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = \alpha T^{\mu\nu}$

$D_\mu G^{\mu\nu} = \alpha D_\mu T^{\mu\nu} = 0$

~~free~~ empty matter $\Rightarrow \quad G^{\mu\nu} = 0 = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R$

$$g_{\mu\nu} G^{\mu\nu} = R - \frac{4}{2} R = 0 \Rightarrow R = 0 \Rightarrow R^{\mu\nu} = 0$$

$$g_{\mu\nu} G^{\mu\nu} = \alpha g_{\mu\nu} T^{\mu\nu} = g_{\mu\nu}(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R)$$

$$R - \frac{4}{2} R = \alpha T \Rightarrow R = -\alpha T$$

$$\Rightarrow \boxed{\begin{array}{l} R^{\mu\nu} = \alpha\left(T^{\mu\nu} - \frac{1}{2} g^{\mu\nu} T\right) \\ T = T^\mu{}_\mu . \end{array}}$$

Action for Einstein's equation

The invariant volume in a curved manifold is $\int d^4x\sqrt{-g}$

We need a scalar Lagrangian density based on metric.

If up to the order of qudratic terms, the only choice is $R$ itself. Then up to a constant, we have

$$S_E = \beta \int d^4x \sqrt{-g}\; g^{\mu\nu} R_{\mu\nu} \; .$$

We consider the variation of metric under,

$$g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$$

$$\lim_{x\to\infty} g_{\mu\nu} \to \eta_{\mu\nu}$$

$$\lim_{x\to\infty} \delta g_{\mu\nu} \to 0$$

Such that $\delta S_E = 0$.

$$g^{\mu\nu} g_{\nu\lambda} = \delta^\mu{}_\lambda \Rightarrow \delta g^{\mu\nu} g_{\nu\lambda} + g^{\mu\nu}\delta g_{\nu\lambda} = 0$$

$$\delta g^{\mu\nu} = - g^{\mu\delta}\delta g_{\delta\lambda}\, g^{\lambda\nu}$$

$$\delta(\sqrt{-g}) = \frac{1}{2\sqrt{-g}}\delta(-g) = \frac{-1}{2\sqrt{-g}}\,\delta e^{\mathrm{tr}\ln g} \qquad = -\frac{\sqrt{-g}}{2} g_{\mu\nu}\delta g^{\nu\mu}$$

$$= \frac{-1}{2\sqrt{-g}} \cdot g\, \mathrm{tr}[g^{-1}\delta g] = \frac{-g}{2\sqrt{-g}} g^{\mu\nu}\delta g_{\nu\mu} = \frac{\sqrt{-g}}{2} g^{\mu\nu}\delta g_{\nu\mu}$$

$$\Gamma^{\rho}_{\mu\nu} = \tfrac{1}{2} g^{\rho\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu})$$

$$\delta\Gamma^{\rho}_{\mu\nu} = \tfrac{1}{2}\,\delta g^{\rho\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu})$$
$$+ \tfrac{1}{2} g^{\rho\sigma}(\partial_\mu \delta g_{\nu\sigma} + \partial_\nu \delta g_{\sigma\mu} - \partial_\sigma \delta g_{\mu\nu})$$

The first line $-\tfrac{1}{2} g^{\rho\alpha}\delta g_{\alpha\beta}\, g^{\beta\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu})$

$$= -g^{\rho\alpha}\delta g_{\alpha\beta}\,\Gamma^{\beta}_{\mu\nu}$$

The 2nd line

$$D_\mu \delta g_{\nu\sigma} = \partial_\mu \delta g_{\nu\sigma} - \Gamma^{\beta}_{\mu\nu}\delta g_{\beta\sigma} - \Gamma^{\beta}_{\mu\sigma}\delta g_{\nu\beta}$$

$$D_\nu \delta g_{\sigma\mu} = \partial_\nu \delta g_{\sigma\mu} - \Gamma^{\beta}_{\nu\sigma}\delta g_{\beta\mu} - \Gamma^{\beta}_{\nu\mu}\delta g_{\sigma\beta}$$

$$D_\sigma \delta g_{\mu\nu} = \partial_\sigma \delta g_{\mu\nu} - \Gamma^{\beta}_{\sigma\mu}\delta g_{\beta\nu} - \Gamma^{\beta}_{\sigma\nu}\delta g_{\mu\beta}$$

$$\Rightarrow D_\mu \delta g_{\nu\sigma} + D_\nu \delta g_{\sigma\mu} - D_\sigma \delta g_{\mu\nu}$$
$$= (\partial_\mu \delta g_{\nu\sigma} + \partial_\nu \delta g_{\sigma\mu} - \partial_\sigma \delta g_{\mu\nu}) - 2\Gamma^{\beta}_{\mu\nu}\delta g_{\beta\sigma}$$

Then $\frac{1}{2} g^{\rho\sigma}(\partial_\mu \delta g_{\nu\sigma} + \partial_\nu \delta g_{\sigma\mu} - \partial_\sigma \delta g_{\mu\nu})$

$$= \tfrac{1}{2} g^{\rho\sigma}(D_\mu \delta g_{\nu\sigma} + D_\nu \delta g_{\sigma\mu} - D_\sigma \delta g_{\mu\nu}) + g^{\rho\sigma}\Gamma^{\beta}_{\mu\nu}\delta g_{\beta\sigma}$$

$$\Rightarrow \delta\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}(D_\mu \delta g_{\nu\sigma} + D_\nu \delta g_{\sigma\mu} - D_\sigma \delta g_{\mu\nu})$$

hence although $\Gamma^{\rho}_{\mu\nu}$ is not a tensor $\therefore$ $\delta\Gamma^{\rho}_{\mu\nu}$ is a tensor.

$$R^{\mu}{}_{\nu\lambda\rho} = \partial_\lambda \Gamma^{\mu}_{\nu\rho} - \partial_\rho \Gamma^{\mu}_{\nu\lambda} + \Gamma^{\mu}_{\lambda\sigma}\Gamma^{\sigma}_{\nu\rho} - \Gamma^{\mu}_{\rho\sigma}\Gamma^{\sigma}_{\nu\lambda}$$

$$R_{\nu\rho} = R^{\mu}{}_{\nu\mu\rho} = \partial_\mu \Gamma^{\mu}_{\nu\rho} - \partial_\rho \Gamma_{\nu}{}^{\mu}{}_{\mu} + \Gamma^{\mu}_{\mu\sigma}\Gamma^{\sigma}_{\nu\rho} - \Gamma^{\mu}_{\rho\sigma}\Gamma^{\sigma}_{\nu\mu}$$

$$\delta R_{\nu\rho} = \partial_\mu \delta\Gamma^{\mu}_{\nu\rho} - \partial_\rho \delta\Gamma^{\mu}_{\nu\mu} + \delta\Gamma^{\mu}_{\mu\sigma}\Gamma^{\sigma}_{\nu\rho} + \Gamma^{\mu}_{\mu\sigma}\delta\Gamma^{\sigma}_{\nu\rho} - \delta\Gamma^{\mu}_{\rho\sigma}\Gamma^{\sigma}_{\nu\mu} - \Gamma^{\mu}_{\rho\sigma}\delta\Gamma^{\sigma}_{\nu\mu}$$

$$= \left(\partial_\mu \delta\Gamma^{\mu}_{\nu\rho} + \Gamma^{\mu}_{\mu\sigma}\delta\Gamma^{\sigma}_{\nu\rho} - \Gamma^{\sigma}_{\mu\nu}\delta\Gamma^{\mu}_{\sigma\rho} - \cancel{\Gamma^{\sigma}_{\mu\rho}\delta\Gamma^{\mu}_{\nu\sigma}}\right)$$

$$- \left(\partial_\rho \delta\Gamma^{\mu}_{\nu\mu} + \Gamma^{\mu}_{\rho\sigma}\delta\Gamma^{\sigma}_{\nu\mu} - \Gamma^{\delta}_{\rho\nu}\delta\Gamma^{\mu}_{\delta\mu} - \cancel{\Gamma^{\delta}_{\rho\mu}\delta\Gamma^{\mu}_{\nu\delta}}\right)$$

$$= D_\mu(\delta\Gamma^{\mu}_{\nu\rho}) - D_\rho(\delta\Gamma^{\mu}_{\nu\rho})$$

$$\Rightarrow \quad \delta R_{\mu\nu} = D_\lambda \delta\Gamma^{\lambda}_{\mu\nu} - D_\nu \delta\Gamma^{\lambda}_{\mu\lambda}$$

$$R = g^{\mu\nu} R_{\mu\nu}$$

$$\delta R = \delta g^{\mu\nu} R_{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}$$

$$= -g^{\mu\lambda} g^{\rho\nu}\delta g_{\lambda\rho} R_{\mu\nu} + g^{\mu\nu}\left(D_\lambda \delta\Gamma^{\lambda}_{\mu\nu} - D_\nu \delta\Gamma^{\lambda}_{\mu\lambda}\right)$$

$$= -\delta g_{\lambda\rho} R^{\lambda\rho} + D_\lambda\left(g^{\mu\nu}\delta\Gamma^{\lambda}_{\mu\nu}\right) - D_\nu\left(g^{\mu\nu}\delta\Gamma^{\lambda}_{\mu\lambda}\right)$$

$$D_\lambda\left(g^{\mu\nu}\delta\Gamma^{\lambda}_{\mu\nu}\right) = \partial_\lambda\left(g^{\mu\nu}\delta\Gamma^{\lambda}_{\mu\nu}\right) + \Gamma^{\lambda}_{\lambda\delta}\left(g^{\mu\nu}\delta\Gamma_{\mu}{}^{\delta}{}_{\nu}\right)$$

$$\Gamma^{\lambda}_{\lambda\delta} = \frac{1}{2} g^{\lambda\gamma}\left(\cancel{\partial_\lambda g_{\delta\gamma}} + \partial_\delta g_{\gamma\lambda} - \cancel{\partial_\gamma g_{\lambda\delta}}\right)$$

$$\Gamma^{\lambda}_{\lambda\delta} = \frac{1}{2}\,\overset{\text{tr}}{}(g^{-1}\partial_\delta g) = \frac{1}{2}\partial_\delta \,\mathrm{tr}\ln \overset{\text{matrix}}{g} = \frac{1}{2}\partial_\delta \ln\det g$$

$$= \frac{1}{2}\partial_\delta [\ln(-\det g)] = \partial_\delta \ln\sqrt{-\det g}$$

$$= \frac{1}{\sqrt{-\det g}}\partial_\delta\sqrt{-\det g} \xrightarrow{\text{abbreviation}} \frac{1}{\sqrt{-g}}\partial_\delta\sqrt{-g}$$

$$D_\lambda(g^{\mu\nu}\delta\Gamma^{\lambda}_{\mu\nu}) = \partial_\lambda(g^{\mu\nu}\delta\Gamma^{\lambda}_{\mu\nu}) + \frac{1}{\sqrt{-g}}\partial_\lambda(\sqrt{-g})(g^{\mu\nu}\delta\Gamma^{\lambda}_{\mu\nu})$$

$$= \frac{1}{\sqrt{-g}}\partial_\nu(\sqrt{-g}\,\delta\Gamma^{\lambda}_{\mu\nu}\, g^{\mu\nu})$$

similarly

$$\delta R = -\delta g_{\mu\nu}R^{\mu\nu} + \frac{1}{\sqrt{-g}}\partial_\nu(\sqrt{-g}\, g^{\mu\nu}\delta\Gamma^{\lambda}_{\mu\nu})$$

$$-\frac{1}{\sqrt{-g}}\partial_\nu(\sqrt{-g}\, g^{\mu\nu}\Gamma^{\lambda}_{\mu\lambda})$$

$$\Rightarrow \delta S_E = \beta\int d^4x\ (\delta\sqrt{-g})R + \sqrt{-g}\,\delta R$$

$$= \beta\int d^4x\ \Big(\frac{1}{2}\sqrt{-g}\ g^{\mu\nu}\delta g_{\mu\nu} R$$

total divergence

$$+\sqrt{-g}\ \Big(-\delta g_{\mu\nu}R^{\mu\nu} + \frac{1}{\sqrt{-g}}\partial_\nu\big(\sqrt{-g}\big(g^{\mu\nu}\delta\Gamma^{\lambda}_{\mu\nu} - g^{\mu\nu}\Gamma^{\lambda}_{\mu\lambda}\big)\Big)\Big]$$

$$= \beta\int d^4x\ \sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}R - R^{\mu\nu}\right)\delta g_{\mu\nu}$$

$$\delta S_E = 0 \Rightarrow R^{\mu\nu} - \tfrac{1}{2} g^{\mu\nu} R = G^{\mu\nu} = 0$$

* In the presence of matter

$$S = S_E + S_{matter}$$

$$= \beta \int d^4x \sqrt{-g}\, R + \int d^4x \sqrt{-g}\, \mathcal{L}_{matter}$$

$$\delta S = \beta \int d^4x \sqrt{-g} \left( \tfrac{1}{2} g^{\mu\nu} R - R^{\mu\nu} \right) + \sqrt{-g} \left( \tfrac{1}{2} \delta g_{\mu\nu} T^{\mu\nu}_{matter} \right)$$

we assume $\delta S_{matter} = \tfrac{1}{2} \int d^4x \sqrt{-g}\, \delta g_{\mu\nu} T^{\mu\nu}$

then $\delta S = \int d^4x \sqrt{-g}\, \delta g_{\mu\nu} \left[ \beta \left( \tfrac{1}{2} g^{\mu\nu} R - R^{\mu\nu} \right) + \tfrac{1}{2} T^{\mu\nu}_{matter} \right)$

$$\Rightarrow R^{\mu\nu} - \tfrac{1}{2} g^{\mu\nu} R = G^{\mu\nu} = \frac{1}{2\beta} T^{\mu\nu}_{matter}$$

$$G^{\mu\nu} = R^{\mu\nu} - \tfrac{1}{2} R g^{\mu\nu} = 8\pi G_N T^{\mu\nu}_{matter}$$

$$\beta = \frac{1}{16\pi G_N} \Rightarrow$$

planck length

$$\Delta x \sim \frac{\hbar}{p} + \frac{GM}{c^2}$$

$$= \frac{\hbar c}{E} + \frac{GE}{c^4} \geq 2\sqrt{\frac{\hbar G}{c^3}}$$

$$l_p^2 = \frac{\hbar G}{c^3} \Rightarrow \frac{c^3}{G} = \hbar l_p^{-2}$$

or $\boxed{G = \frac{c^3}{\hbar} l_p^2}$

If adding back "c", we have.

$$S_E = \frac{c^3}{16\pi G_N} \int d^4x \sqrt{-g}\, R$$

$$\Rightarrow \frac{S_E}{\hbar} = \frac{l_p^{-2}}{16\pi} \int d^4x \sqrt{-g}\, R$$