Lecture 9 Schwarzschild Solution

- Schwarzschild solution

$$d\tau^2 = \left(1 - \frac{2G_N M}{c^2 r}\right) dt^2 - \left(1 - \frac{2G_N M}{c^2 r}\right)^{-1} dr^2$$
$$- r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right)$$

- 3D space of constant curvature

§ line element with spherical symmetry

$$d\tau^2 = -g_{\mu\nu}\, dx^\mu dx^\nu$$
$$= A(r)(dt)^2 - [B(r)\, dr^2 + C(r)\, r^2 d\theta^2 + D(r)\, r^2 \sin^2\theta\, d\phi^2]$$

① Static solution $d\tau^2$ should be invariant as $dt \to -dt$

hence terms linear to dt should not appear

② isotropy: $d\theta \to -d\theta$, $d\varphi \to -d\varphi$, $d\tau^2$ should be invariant

$\Rightarrow$ the metric has to be diagonal

The solid angle $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$

$$\Rightarrow d\tau^2 = A(r)(dt)^2 - B(r)\, dr^2 - C(r)\, r^2 d\Omega^2$$

C(r) can be absorbed $\tilde{r} = [C(r)]^{1/2}\, r$

$$d\tilde{r} = dr\left[(C(r))^{1/2} + \frac{1}{2}\frac{rC'(r)}{[C(r)]^{1/2}}\right]$$

or $dr = f(\tilde{r})\, d\tilde{r}$

Then $d\tau^2 = \tilde{A}(\tilde{r})\, dt^2 - \tilde{B}(\tilde{r})\, d\tilde{r}^2 - \tilde{r}^2(d\theta^2 + \sin^2\theta\, d\phi^2)$

or simply $d\tau^2 = A(r)dt^2 - \underline{B(r)\,dr^2} - r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$

gravity is along radial direction

§ connection

$$g_{00} = g_{tt} = -A(r) \qquad g_{22} = g_{\theta\theta} = r^2$$

$$g_{11} = g_{rr} = B(r) \qquad g_{33} = g_{\phi\phi} = r^2 \sin^2\theta$$

$$\Rightarrow \quad g^{00} = g^{tt} = -1/A(r) \qquad g^{11} = 1/B(r), \; g^{22} = r^{-2}, \; g^{33} = r^{-2}\sin^{-2}\theta$$

$$\Gamma^{\mu}_{\nu\lambda} = 1/2\, g^{\mu\rho} (\partial_\nu g_{\lambda\rho} + \partial_\lambda g_{\rho\nu} - \partial_\rho g_{\nu\lambda})$$

The geodesic Eq

$$\frac{d^2x^\nu}{dz^2} + \Gamma^{\nu}_{\lambda\rho} \frac{dx^\lambda}{dz} \frac{dx^\rho}{dz} = 0$$

$$d^2x^\nu/dz^2 + 1/2\, g^{\nu\mu} (\partial_\lambda g_{\rho\mu} + \partial_\rho g_{\mu\lambda} - \partial_\mu g_{\lambda\rho}) \frac{dx^\lambda}{dz} \frac{dx^\rho}{dz} = 0$$

$$g_{\mu\nu} \frac{d^2x^\nu}{dz^2} + 1/2 (\partial_\lambda g_{\rho\mu} + \partial_\rho g_{\mu\lambda} - \partial_\mu g_{\lambda\rho}) \frac{dx^\lambda}{dz} \frac{dx^\rho}{dz} = 0$$

For $\mu = 0 \Rightarrow$

$$g_{00} \frac{d^2x^0}{dz^2} + 1/2 (\partial_\lambda g_{\rho 0} + \partial_\rho g_{0\lambda} - \partial_0 g_{\lambda\rho}) \frac{dx^\lambda}{dz} \frac{dx^\rho}{dz} = 0$$

$$-A(r) \frac{d^2x^0}{dz^2} + \partial_1 g_{00} \frac{dx^0}{dz} \frac{dx^1}{dz} = 0$$

$$\frac{d^2x^0}{dz^2} + \frac{A'(r)}{A(r)} \frac{dx^0}{dz} \frac{dx^1}{dz} = 0 \Rightarrow \boxed{\Gamma^0_{01} = \Gamma^0_{10} = \frac{1}{2} \frac{A'(r)}{A(r)}}$$

For $\mu = 1 \Rightarrow$

$$g_{11} \frac{d^2x^1}{dz^2} + 1/2 (\partial_\lambda g_{\rho 1} + \partial_\rho g_{1\lambda} - \partial_1 g_{\lambda\rho}) \frac{dx^\lambda}{dz} \frac{dx^\rho}{dz} = 0$$

$$B(r) \frac{d^2x^1}{dz^2} + 1/2 (\partial_1 g_{11} \left(\frac{dx^1}{dz}\right)^2 \times 2 - \partial_1 (g_{00}) \left(\frac{dx^0}{dz}\right)^2 - \cdots \; \partial_1 (g_{33}) \left(\frac{dx^3}{dz}\right)^2 = 0$$

$$B(r)\frac{d^2x^1}{d\tau^2} + \frac{1}{2}\left[-\partial_1 g_{00}\left(\frac{dx^0}{d\tau}\right)^2 + \partial_1 g_{11}\left(\frac{dx^1}{d\tau}\right)^2 - \partial_1 g_{22}\left(\frac{dx^2}{d\tau}\right)^2 - \partial_1 g_{33}\left(\frac{dx^3}{d\tau}\right)^2\right] = 0$$

$$\frac{d^2x^1}{d\tau^2} + \frac{1}{2}\frac{A'}{B}\left(\frac{dx^0}{d\tau}\right)^2 + \frac{1}{2}\frac{B'}{B}\left(\frac{dx^1}{d\tau}\right)^2 - \frac{r}{B}\left(\frac{dx^2}{d\tau}\right)^2 - \frac{r}{B}\sin^2\theta\left(\frac{dx^3}{d\tau}\right)^2 = 0$$

$$\Rightarrow \boxed{\begin{cases} \Gamma^1_{00} = \frac{1}{2}\frac{A'}{B}, & \Gamma^1_{11} = \frac{1}{2}B'/B, \\ \Gamma^1_{22} = -r/B, & \Gamma^1_{33} = -\frac{r\sin^2\theta}{B} \end{cases}}$$

(*)

$\mu=2$

$$\Rightarrow g_{22}\frac{d^2x^2}{d\tau^2} + \frac{1}{2}(\partial_\lambda g_{\rho 2} + \partial_\rho g_{2\lambda} - \partial_2 g_{\lambda\rho})\frac{dx^\lambda}{d\tau}\frac{dx^\rho}{d\tau} = 0$$

$$r^2\frac{d^2x^2}{d\tau^2} + \partial_1 g_{22}\frac{dx^1}{d\tau}\frac{dx^2}{d\tau} - \frac{1}{2}\partial_2 g_{33}\left(\frac{dx^3}{d\tau}\right)^2 = 0$$

$$\frac{d^2x^2}{d\tau^2} + \frac{2r}{r^2}\frac{dx^1}{d\tau}\frac{dx^2}{d\tau} - \frac{r^2\sin\theta\cos\theta}{r^2}\left(\frac{dx^3}{d\tau}\right)^2 = 0$$

$$\frac{d^2x^2}{d\tau^2} + \frac{2}{r}\frac{dx^1}{d\tau}\frac{dx^2}{d\tau} - \sin\theta\cos\theta\left(\frac{dx^3}{d\tau}\right)^2 = 0$$

$$\boxed{\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r} \qquad \Gamma^2_{33} = -\sin\theta\cos\theta}$$

$\mu=3$: $\qquad \partial_3 g_{\lambda\rho}=0$

$$g_{33}\frac{d^2x^3}{d\tau^2}+\frac{1}{2}\left(\partial_\lambda g_{\rho 3}+\partial_\rho g_{3\lambda}-\partial_3 g_{\lambda\rho}\right)\frac{dx^\lambda}{d\tau}\frac{dx^\rho}{d\tau}=0$$

$$r^2\sin^2\theta\frac{d^2x^3}{d\tau^2}+\partial_1 g_{33}\frac{dx^3}{d\tau}\frac{dx^1}{d\tau}+\partial_2 g_{33}\frac{dx^3}{d\tau}\frac{dx^2}{d\tau}=0$$

$$r^2\sin^2\theta\frac{d^2x^3}{d\tau^2}+2r\sin^2\theta\frac{dx^1}{d\tau}\frac{dx^3}{d\tau}+2r^2\sin\theta\cos\theta\frac{dx^2}{d\tau}\frac{dx^3}{d\tau}=0$$

$$\frac{d^2x^3}{d\tau^2}+\frac{2}{r}\frac{dx^1}{d\tau}\frac{dx^3}{d\tau}+2\cot\theta\frac{dx^3}{d\tau}=0$$

$$\boxed{\Gamma^3_{13}=\Gamma^3_{31}=\frac{1}{r}\qquad \Gamma^3_{23}=\Gamma^3_{32}=\cot\theta}$$

(*) Solution to the Einstein equation

$$R^\rho{}_{\mu\lambda\nu}=\partial_\lambda\Gamma^\rho_{\mu\nu}-\partial_\nu\Gamma^\rho_{\mu\lambda}+\Gamma^\rho_{\lambda\sigma}\Gamma^\sigma_{\mu\nu}-\Gamma^\rho_{\nu\sigma}\Gamma^\sigma_{\mu\lambda}$$

$$R_{\mu\nu}=R^\lambda{}_{\mu\lambda\nu}=\partial_\lambda\Gamma^\lambda_{\mu\nu}-\partial_\nu\Gamma^\lambda_{\mu\lambda}+\Gamma^\lambda_{\lambda\sigma}\Gamma^\sigma_{\mu\nu}-\Gamma^\lambda_{\nu\sigma}\Gamma^\sigma_{\mu\lambda}$$

$$R_{00}=\partial_\lambda\Gamma^\lambda_{00}-\cancel{\partial_0\Gamma^\lambda_{0\lambda}}+\Gamma^\lambda_{\lambda\sigma}\Gamma^\sigma_{00}-\Gamma^\lambda_{0\sigma}\Gamma^\sigma_{0\lambda}$$

$$=\partial_i\Gamma^i_{00}+\Gamma^\lambda_{\lambda 1}\Gamma^1_{00}-\Gamma^0_{0\sigma}\Gamma^\sigma_{00}-\Gamma^1_{0\sigma}\Gamma^\sigma_{01}$$

$$=\partial_1\Gamma^1_{00}+\left(\Gamma^0_{01}+\Gamma^1_{11}+\Gamma^2_{21}+\Gamma^3_{31}\right)\Gamma^1_{00}-\Gamma^0_{01}\Gamma^1_{00}-\Gamma^1_{00}\Gamma^0_{01}$$

$$=\partial_1\Gamma^1_{00}+\left(-\Gamma^0_{01}+\Gamma^1_{11}+\Gamma^2_{21}+\Gamma^3_{31}\right)\Gamma^1_{00}$$

$$=\frac{1}{2}\frac{\partial}{\partial r}\left(\frac{A'}{B}\right)+\frac{1}{2}\left(\frac{-A'}{A}+\frac{1}{2}\frac{B'}{B}+\frac{1}{r}+\frac{1}{r}\right)\left(\frac{1}{2}\frac{A'}{B}\right)$$

$$\boxed{R_{00}=\frac{1}{2}\frac{\partial}{\partial r}\left(\frac{A'}{B}\right)-\frac{1}{2}\frac{A'}{B}\left(\frac{1}{2}\frac{A'}{A}-\frac{1}{2}\frac{B'}{B}-\frac{2}{r}\right)}$$

$$R_{11} = \partial_\lambda \Gamma^\lambda_{11} - \partial_1 \Gamma^\lambda_{1\lambda} + \Gamma^\lambda_{\lambda\sigma}\Gamma^\sigma_{11} - \Gamma^\lambda_{1\sigma}\Gamma^\sigma_{1\lambda}$$

$$= -\partial_1 (\Gamma^0_{10} + \cancel{\Gamma^1_{11}} + \Gamma^2_{12} + \Gamma^3_{13}) + \partial_1 \cancel{\Gamma^1_{11}}$$

$$+ \Gamma^1_{11} (\Gamma^0_{01} + \cancel{\Gamma^1_{11}} + \Gamma^2_{21} + \Gamma^3_{31})$$

$$- \Gamma^0_{10}\Gamma^0_{10} - \cancel{\Gamma^1_{11}\Gamma^1_{11}} - \Gamma^2_{12}\Gamma^2_{12} - \Gamma^3_{13}\Gamma^3_{13}$$

$$= -\frac{\partial}{\partial r}\left(\frac{1}{2}\frac{A'}{A} + \frac{1}{r} + 1/r \right) + \frac{1}{2}\frac{B'}{B}\left(\frac{1}{2}\frac{A'}{A} + \frac{2}{r} \right)$$

$$- 1/4 \left(\frac{A'}{A}\right)^2 - 2/r^2$$

$$= -\frac{A''}{2A} + 1/2\, A' \frac{A'}{A^2} + \frac{2}{r^2} + 1/4 \frac{A'}{A}\frac{B'}{B} + \frac{B'}{rB} - 1/4\left(\frac{A'}{A}\right)^2 - 2/r^2$$

$$\boxed{R_{11} = -\frac{A''}{2A} + 1/4\, A'/A \left(\frac{A'}{A} + B'/B \right) + B'/rB}$$

$$R_{22} = \partial_\lambda \Gamma^\lambda_{22} - \partial_2 \Gamma^\lambda_{2\lambda} + \Gamma^\lambda_{\lambda\sigma}\Gamma^\sigma_{22} - \Gamma^\lambda_{2\sigma}\Gamma^\sigma_{2\lambda}$$

$$= \partial_1 \Gamma^1_{22} - \partial_2 \Gamma^3_{23} + \Gamma^1_{22}(\Gamma^0_{01} + \Gamma^1_{11} + \cancel{\Gamma^2_{21}} + \Gamma^3_{31})$$

$$- \cancel{\Gamma^1_{22}\Gamma^2_{21}} - \Gamma^2_{21}\Gamma^1_{22} - \Gamma^3_{23}\Gamma^3_{23}$$

$$= -\frac{\partial}{\partial r}\left(\frac{r}{B}\right) - \frac{d}{d\theta}\cot\theta + \left(\frac{-r}{B}\right)\left(\frac{1}{2}\frac{A'}{A} + \frac{1}{2}\frac{B'}{B} + 1/r \cancel{+} 1/r \right)$$

$$- \cot^2\theta$$

$$= 1 - 1/B + \frac{rB'}{B^2} - \frac{1}{2}\frac{r}{B}\left(\frac{A'}{A} + \frac{B'}{B} \right)$$

$$\boxed{R_{22} = 1 - 1/B - \frac{r}{2B}\left(\frac{A'}{A} - \frac{B'}{B} \right)}$$

$$R_{22} = 0 \Rightarrow \frac{1}{B} + \frac{r}{2B}\left(\frac{A'}{A} - \frac{B'}{B}\right) - 1 = 0$$

$$R_{11} = 0 \Rightarrow \frac{A''}{2A} - \frac{1}{4}\frac{A'}{A}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{B'}{rB} = 0 \quad ①$$

$$R_{00} = 0 \Rightarrow \frac{A''}{2B} - \frac{1}{4}\frac{A'}{B}\left(\frac{A'}{A} + \frac{B'}{B}\right) + \frac{A'}{rB} = 0$$

$\frac{B}{A} \times \rightarrow$

$$\rightarrow \frac{A''}{2A} - \frac{1}{4}\frac{A'}{A}\left(\frac{A'}{A} + \frac{B'}{B}\right) + \frac{A'}{Ar} = 0 \quad ②$$

$$① - ② \Rightarrow \frac{1}{r}\left(\frac{A'}{A} + \frac{B'}{B}\right) = 0$$

$$\Rightarrow \frac{d}{dr}(\ln A + \ln B) = 0 \Rightarrow AB = \text{constant}$$

since $\lim_{r\to\infty} A(r) = \lim_{r\to\infty} B(r) = 1 \Rightarrow \boxed{AB = 1}$

plug in $R_{22} = 0 \Rightarrow A + \frac{rA}{2}\left(2\frac{A'}{A}\right) - 1 = 0$

$$A(r) + rA'(r) = 1 \Rightarrow \frac{d(rA(r))}{dr} = 1$$

$$rA(r) = r + \text{cont} \Rightarrow A(r) = 1 + \frac{c}{r}$$

c ← const to be determined

$$\Rightarrow \boxed{\begin{aligned} A(r) &= 1 + c/r \\ B(r) &= (1 + c/r)^{-1} \end{aligned}}$$

$$\boxed{C = -2G_N M}$$

The weak field approximation, $\Rightarrow$ $g_{00} + 1 = \frac{2G_N M}{r}$

$$= -A(r) + 1 = -c/r$$

$$\Rightarrow \boxed{d\tau^2 = \left(1 - \frac{2G_N M}{c^2 r}\right) dt^2 - \left(1 - \frac{2G_N M}{c^2 r}\right)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2\theta \, d\phi^2)}$$

Schwartzchild

At $r = 2G_N M / c^2$, the Schwarzschild metric is singular, $g_{00} = 0$, $g_{rr} \to \infty$.

$G_N / c^2 \simeq 7 \times 10^{-29}$ cm/g $= 7 \times 10^{-28}$ m/kg

$M_{sun} \simeq 2 \times 10^{30}$ kg $\quad M_{earth} \simeq 6 \times 10^{24}$ kg

$\Rightarrow r_{\odot} = 2 G_N M / c^2 \simeq 2 \times 2 \times 7 \times 10^{2}$ m $\simeq 3$ km.

$r_s^{earth} \simeq 1$ cm.

For proton $\quad m \simeq 1\,\text{GeV}/c^2 \simeq 1.6 \times 10^{-27}$ kg

$r_s^{proton} \simeq 2.2 \times 10^{-52}$ cm

but proton radius $= 1.2 \times 10^{-13}$ cm.

- The curvature tensor $R_{\mu\nu}$ is regular at r_s. The singularity of the metric is an artifact of our choice of coordinate. If the mass is compressed so much inside r_s, then we arrive at a black hole. r_s is referred as the black hole horizon.

- Example (3D ~~f~~ space of constant curvature)

$$R_{\mu\nu\lambda\rho} = k\,(g_{\mu\lambda}\, g_{\nu\rho} - g_{\mu\rho}\, g_{\nu\lambda})$$

Consider an isotropic 3D space, $d\tau^2 = A(r)\, dr^2 + r^2 d\Omega^2$

$$g_{11} = A(r), \quad g_{22} = r^2 \quad g_{33} = r^2 \sin^2\theta$$

According to

$$\frac{d^2 x^\nu}{d\tau^2} + \Gamma^\nu_{\lambda\rho} \frac{dx^\lambda}{d\tau}\frac{dx^\rho}{d\tau} = 0$$

$$g_{\mu\nu} \frac{d^2 x^\nu}{d\tau^2} + 1/2\, (\partial_\lambda g_{\rho\mu} + \partial_\rho g_{\mu\lambda} - \partial_\mu g_{\lambda\rho}) \frac{dx^\lambda}{d\tau}\frac{dx^\rho}{d\tau} = 0$$

$$\mu = 1 \Rightarrow A(r) \frac{d^2 x^1}{d\tau^2} + 1/2\, (\partial_\lambda g_{\rho 1} + \partial_\rho g_{1\lambda} - \partial_1 g_{\lambda\rho}) \frac{dx^\lambda}{d\tau}\frac{dx^\rho}{d\tau} = 0$$

$$A(r) \frac{d^2 x^1}{d\tau^2} + \partial_1 g_{11} \left(\frac{dx^1}{d\tau}\right)^2 - 1/2 \left[(\partial_1 g_{11}) \left(\frac{dx^1}{d\tau}\right)^2 + \partial_1 g_{22} \left(\frac{dx^2}{d\tau}\right)^2 + \partial_1 g_{33} \left(\frac{dx^3}{d\tau}\right)^2 \right] = 0$$

$$\frac{d^2 x^1}{d\tau^2} + \frac{1}{2}\frac{A'}{A} \left(\frac{dx^1}{d\tau}\right)^2 - 1/2 \left(\frac{2r}{A}\right) \left(\frac{dx^2}{d\tau}\right)^2 - 1/2 \left(\frac{2r \sin^2\theta}{A}\right) \left(\frac{dx^3}{d\tau}\right)^2 = 0$$

$$\frac{d^2 x^1}{d\tau^2} + \frac{1}{2}\frac{A'}{A} \left(\frac{dx^1}{d\tau}\right)^2 - \frac{r}{A} \left(\frac{dx^2}{d\tau}\right)^2 - \frac{r \sin^2\theta}{A} \left(\frac{dx^3}{d\tau}\right)^2 = 0$$

$$\boxed{\Gamma^1_{11} = \frac{A'}{2A}, \quad \Gamma^1_{22} = -\frac{r}{A}, \quad \Gamma^1_{33} = \frac{r}{A}\sin^2\theta}$$

$\mu = 2$

$$r^2 \frac{d^2 x^2}{d\tau^2} + \frac{1}{2}\left(\partial_\lambda g_{\rho 2} + \partial_\rho g_{2\lambda} - \partial_2 g_{\lambda\rho}\right) \frac{dx^\lambda}{d\tau}\frac{dx^\rho}{d\tau} = 0$$

$$r^2 \frac{d^2 x^2}{d\tau^2} + 2r\left(\frac{dx^2}{d\tau}\right)\left(\frac{dx^1}{d\tau}\right) - \frac{1}{2}\frac{d}{d\theta}\left(r^2 \sin^2\theta\right)\frac{dx^3}{d\tau}\frac{dx^3}{d\tau} = 0$$

$$\frac{d^2 x^2}{d\tau^2} + \frac{2}{r}\left(\frac{dx^1}{d\tau}\right)\left(\frac{dx^2}{d\tau}\right) - \sin\theta\cos\theta\left(\frac{dx^3}{d\tau}\right)^2 = 0$$

$$\boxed{\Gamma^2_{12} = \Gamma^2_{21} = 1/r, \qquad \Gamma^2_{33} = -\sin\theta\cos\theta}$$

$\mu = 3$

$$r^2 \sin^2\theta \frac{d^2 x^3}{d\tau^2} + \frac{1}{2}\left(\partial_\lambda g_{\rho 3} + \partial_\rho g_{3\lambda} - \partial_3 g_{\lambda\rho}\right)\frac{dx^\lambda}{d\tau}\frac{dx^\rho}{d\tau} = 0$$

$$r^2 \sin^2\theta \frac{d^2 x^3}{d\tau^2} + \left(2r\sin^2\theta \frac{dx^1}{d\tau}\frac{dx^3}{d\tau} + 2r^2 \sin\theta\cos\theta \left(\frac{dx^3}{d\tau}\right)\left(\frac{dx^2}{d\tau}\right) = 0\right.$$

$$\cancel{- \frac{1}{2} r^2 \, 2\sin\theta\cos\theta \left(\frac{dx^3}{d\tau}\right)^2 = 0}$$

$$\frac{d^2 x^3}{d\tau^2} + \frac{2}{r}\frac{dx^1}{d\tau}\frac{dx^3}{d\tau} + 2\cot\theta\left(\frac{dx^3}{d\tau}\right)^2\left(\frac{dx^2}{d\tau}\right) = 0$$

$$\boxed{\Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r}, \qquad \Gamma^3_{23} = \Gamma^3_{32} = \cot\theta}$$

$$R_{\mu\nu\lambda\rho} = g_{\mu\sigma}\left(\partial_\lambda \Gamma^\sigma_{\nu\rho} - \partial_\rho \Gamma^\sigma_{\nu\lambda} + \Gamma^\beta_{\nu\rho}\Gamma^\sigma_{\beta\lambda} - \Gamma^\beta_{\nu\lambda}\Gamma^\sigma_{\beta\rho}\right)$$

$\mu = \lambda = 1, \ \nu = \rho = 2$

$$R_{1212} = g_{1\sigma}\left(\partial_1 \Gamma^\sigma_{22} - \partial_2 \Gamma^\sigma_{21} + \Gamma^\beta_{22}\Gamma^\sigma_{\beta 1} - \Gamma^\beta_{21}\Gamma^\sigma_{\beta 2}\right)$$

$$= A\left[\partial_r\left(\frac{-r}{A}\right) + \Gamma^1_{22}\Gamma^1_{11} - \Gamma^2_{21}\Gamma^1_{22}\right] = -A\,\partial_r\left(\frac{r}{A}\right) - A\frac{r}{A}\frac{A'}{2A} - A\frac{1}{r}\left(\frac{-r}{A}\right)$$

$$R_{1212} = -1 + A\frac{rA'}{A^2} - \frac{rA'}{2A} + 1 = \frac{r}{2A}A'$$

$$R_{1212} = K(g_{11}g_{22} - g_{12}g_{21}) = KAr^2$$

$$\Rightarrow \quad \frac{rA'}{2A} = KAr^2 \quad \Rightarrow \quad \frac{A'}{A^2} = 2Kr \quad \Rightarrow \quad \left(\frac{1}{A}\right)' = -2Kr$$

$$\frac{1}{A} = \underset{\text{const}}{} - Kr^2 \quad \Rightarrow \quad A = \frac{1}{\text{const} - Kr^2}$$

when $K \to 0$, we should recover the ordinary spherical coordinate $A = 1 \Rightarrow$ const $= 1$.

$$A = \frac{1}{1-Kr^2} \quad \Rightarrow \quad \boxed{d\tau^2 = \frac{dr^2}{1-Kr^2} + r^2 d\Omega^2}$$

If $K > 0$, and $Kr^2 < 1$ we define

$$\frac{(dr)^2}{1-Kr^2} = (d\rho)^2 \quad \Rightarrow \quad \frac{1}{\sqrt{K}}\frac{dr'}{\sqrt{1-r'^2}} = d\rho$$

$$\Rightarrow \quad \sin^{-1} r' = \sqrt{K}\, d\rho \quad \Rightarrow \quad \sqrt{K}\, r = \sin\sqrt{K}\rho$$

$$r = \frac{\sin\sqrt{K}\rho}{\sqrt{K}} \quad \Rightarrow \quad d\tau^2 = d\rho^2 + \frac{1}{K}\sin^2\sqrt{K}\rho \; d\Omega^2$$

If $K < 0$, $K = -|K| \Rightarrow r = \sinh\sqrt{|K|}\rho \,/\, \sqrt{|K|}$ $\quad d\rho^2 = \frac{dr^2}{1+|K|r^2}$

$$\boxed{d\tau^2 = d\rho^2 + \frac{1}{|K|}\sinh^2(\sqrt{|K|}\rho)\, d\Omega^2}$$

Properties of Schwarzschild Solution

$$d\tau^2 = \left(1 - \frac{2G_N M}{c^2 r}\right) dt^2 - \left(1 - \frac{2G_N M}{c^2 r}\right) dr^2 - r^2 (d\theta^2 + \sin^2\theta \, d\phi^2)$$

$$dR = \left(1 - \frac{2G_N M}{c^2 r}\right)^{-1/2} dr \geq dr \quad \text{at a fixed time}$$

Radial increment dR on a curved space compared with its projection dr

For a clock at rest, the proper time

$$d\tau = \left(1 - \frac{2G_N M}{c^2 r}\right)^{1/2} dt$$

c.f. special relativity

$$d\tau_{SR} = \left(1 - \left(\frac{v}{c}\right)^2\right)^{1/2} dt$$

$$dL_{SR} = \left[1 - \left(\frac{v}{c}\right)^2\right]^{-1/2} dl$$

Analogy with unit sphere

$$ds^2 = d\rho^2 + \sin^2\rho \, d\varphi^2$$

if $\sin\rho \to \rho$, then it becomes the flat 2D space