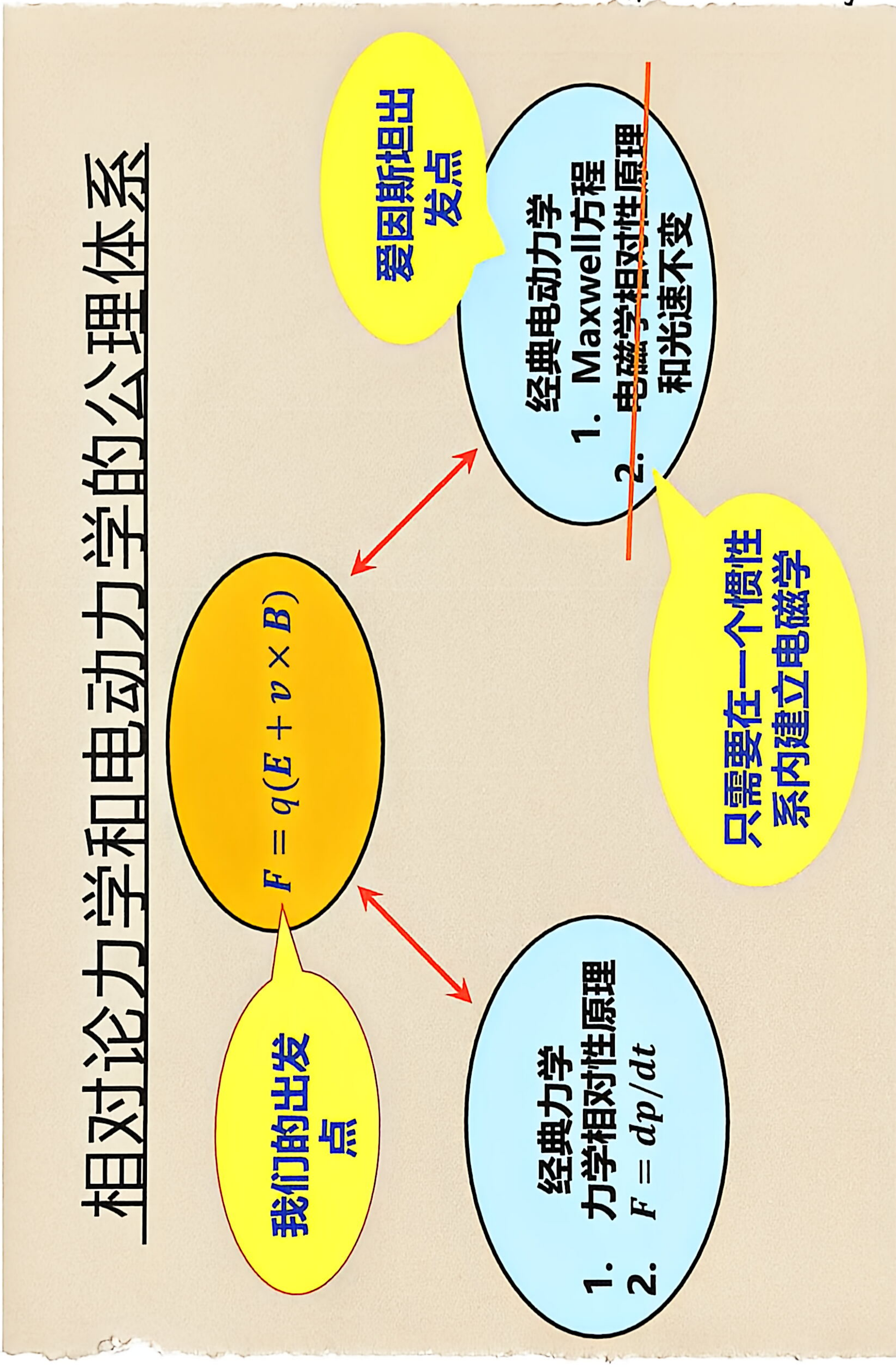
Lect 1 Rethink relativity

— The existance of a universal upper limit of speed



Galilean's principle of relativity

- **Post 1:** Laws of mechanical motions are invariant in all inertial frames.

With the ship standing still, observe carefully how the little animals fly with equal speed to all sides of the cabin. The fish swim indifferently in all directions….

Have the ship proceed with any speed you like, so long as the motion is uniform ….. You will discover not the least change in all the effects named, nor could you tell from any of them whether the ship was moving or standing still….

the butterflies and flies will continue their flights indifferently toward every side, nor will it ever happen that they are concentrated toward the stern, as if tired out from keeping up with the course of the ship

Dialogue Concerning the Two Chief World Systems, --- Galileo

Galileo (1564 – 1642)

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Fundamentals of kinematics

- **Axiom 1 (symmetries):** In every inertial frame, space is homogeneous and isotropic, and time is homogeneous.

- **Axiom 2 (Law of inertia):** If there is no force exerted on a particle in an inertial frame, it keeps the state of a uniform rectilinear motion. If there is a net force on a particle, it accelerates.

- **Corollary 1:** A uniform rectilinear motion observed in one inertial frame is also uniform rectilinear when observed in another inertial frame.

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History overview

① Oerested experiment (1820)

Biot - Savart's law, Ampère's law

② Weber – Kohlrausch experiment (1855)

The electrostatic force $F = \frac{Q^2}{r^2}$

The electromagnetic force $F = \frac{2I^2 l'}{r c^2}$

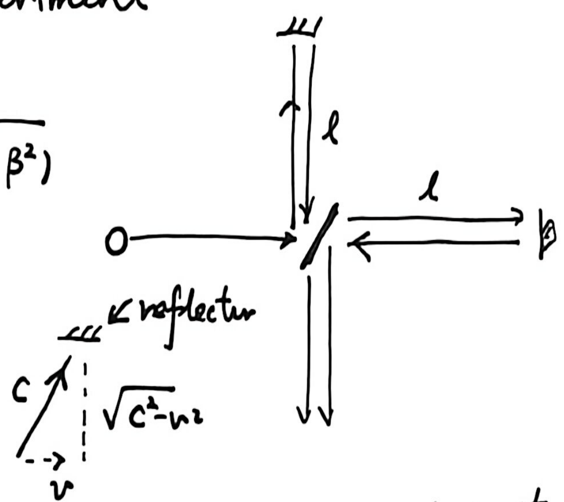
C - experiment value 4.4×10^8 m/s (not accurate)

③ Faraday's electromagnetic induction
Maxwell's displacement current
} not necessary for establishing relativity

④ Michelson - Morley experiment

$$t_1 = \frac{l}{c+v} + \frac{l}{c-v} = \frac{2l}{c(1-\beta^2)}$$

$$\simeq \frac{2l}{c}(1+\beta^2+\cdots), \quad \beta = \frac{v}{c}.$$



$$t_2 = \frac{2l}{\sqrt{c^2-v^2}}$$

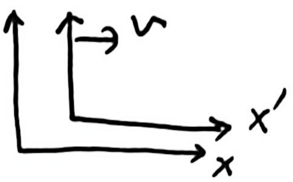
$$= \frac{2l}{c}(1+\tfrac{1}{2}\beta^2+\cdots)$$

$$\Delta t = t_1 - t_2 = \frac{l}{c}\beta^2$$ ← but Δt was not observed!

§ Inconsistency of Galilean transformation and EM

We can define two different types of Galilean 4-vectors

① $\begin{cases} x' = x - vt \\ t' = t \end{cases}$ ② $\begin{cases} p' = p - vm \\ m' = m \end{cases}$



③ $\begin{cases} k' = k \\ \omega' = \omega - vk \end{cases}$ ④ $\begin{cases} \frac{A'}{c} = A/c \\ \varphi' = \varphi - \frac{v}{c} \cdot A \end{cases}$

A: The inner product between ① and ③ $\Rightarrow$ $\boxed{kx - \omega t}$,

The phase change, which is an invariant independent on reference frames in both relativistic and non-relativistic physics.

B: The inner product ① and ④ $\Rightarrow$

$$q\left(\frac{\vec{A}}{c} \cdot d\vec{x} - \varphi\, dt\right) \Rightarrow$$ electron-magnetism phase in quantum mechanics

C: ② · ④ $\Rightarrow$

$$\frac{1}{c}\vec{p} \cdot \vec{A} - m\varphi = m\left(\frac{\vec{v}}{c} \cdot \vec{A} - \varphi\right)$$ ← proportional to Langrangian of charged particle to EM field

Classic Lagrangian of a charged particle

$$\mathcal{L} = \tfrac{1}{2} m \dot{x}^2 + \left(\frac{\vec{j}}{c}\cdot\vec{A} - q\varphi\right) \quad \text{where } \vec{j} = q\dot{x}$$

$$\begin{cases} j' = j - qv \\ q' = q \end{cases} \Rightarrow \frac{1}{c}\vec{j}\cdot\vec{A} - q\varphi \text{ is a Galilean invariant}$$

$$\mathcal{L}' = \tfrac{1}{2} m (\dot{x}' + v)^2 + \left(\vec{j}'\cdot\vec{A}'\cdot\frac{1}{c} - q\varphi'\right)$$

$$= \mathcal{L} + mv\, dx'/dt \leftarrow \text{total derivative}$$

$$\rightarrow \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}}\right) = \frac{\partial\mathcal{L}}{\partial x} \Rightarrow m\ddot{x} = q\left(-\frac{1}{c}\frac{\partial\vec{A}}{\partial t} - \nabla\varphi\right) + \frac{q}{c}\left[\dot{x}\times(\nabla\times\vec{A})\right]$$

$$\vec{F} = q\left(\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right)$$

However, $\begin{cases} A'/c = A/c \\ \varphi' = \varphi - \frac{v}{c}\cdot A \end{cases} \Rightarrow \begin{cases} \vec{B}' = \vec{B} \\ \vec{E}' = \vec{E} - \frac{\vec{v}}{c}\times\vec{B} - (\vec{v}\cdot\nabla)\vec{A} \end{cases}$

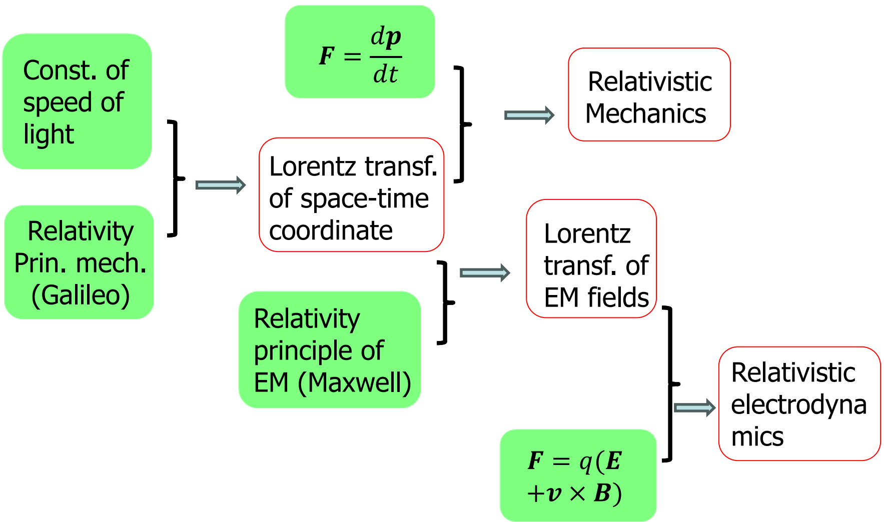
This is inconsistent even with the Maxwell's equation for the steady field:

$$\nabla\times\vec{B} = \frac{4\pi}{c}\vec{j}$$

Ampère's law

Even without referring to EM waves, we have already found the inconsistence of classic E&M with Galilean transform.

Einstein's Construction of Special Relativity



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Postulates in Special Relativity

- **Not self-evident**:

1. Light speed invariance is against daily experience of velocity addition
2. Historically, there were long-term debates on whether Maxwell's Eqs are frame-independent or not.

Constancy of light speed and the invariance of Maxwell's Eqs. are like Pythagorean Theorem, though true, are not suitable for being treated as postulates.

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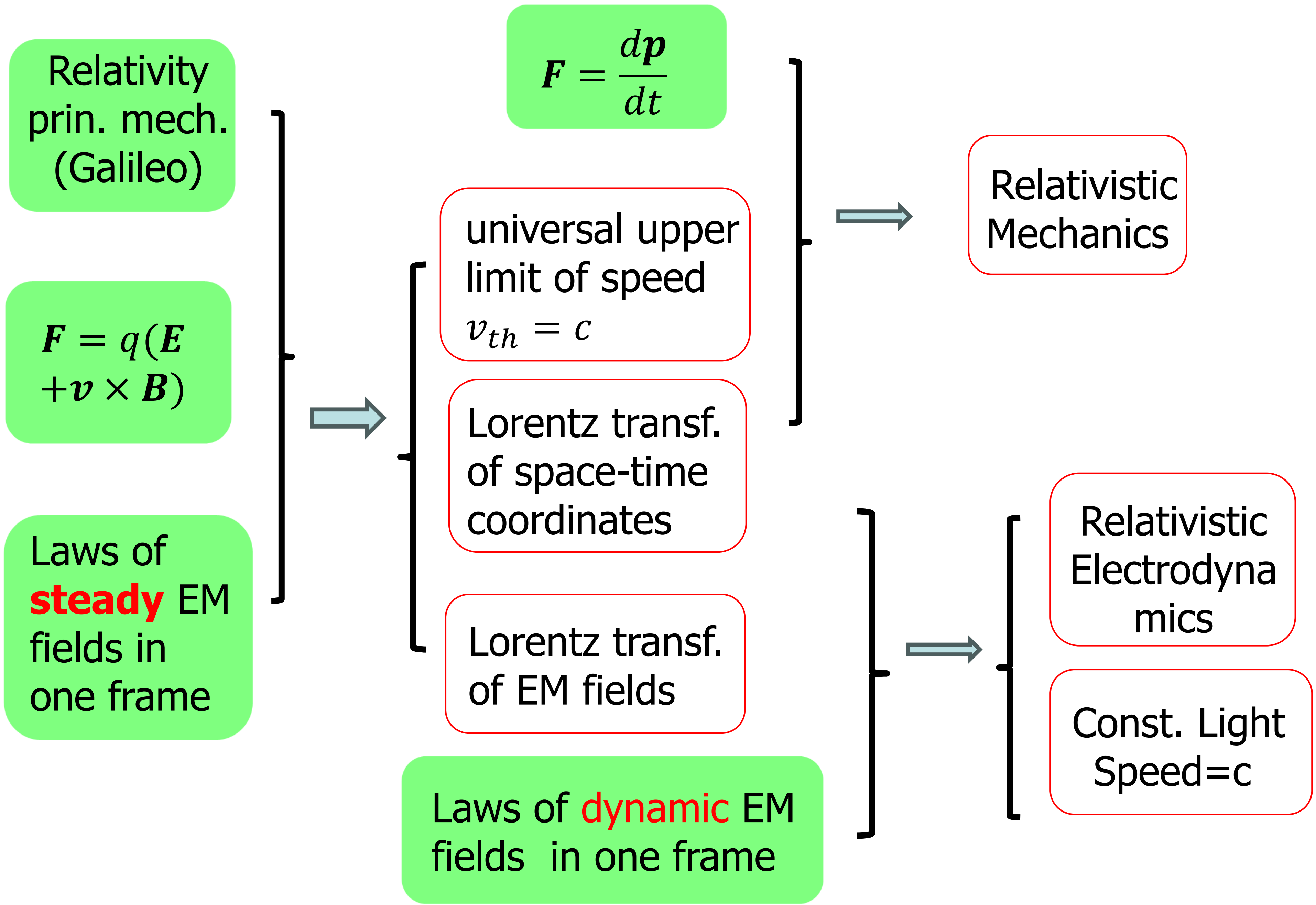
Our Construction of Special Relativity

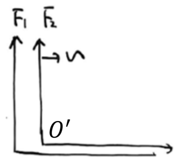


Clock synchronization without light

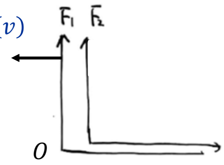
- 滴滴哒--, 刚才最后一响，北京时间12点整。

- F_2 moves at the velocity v relative to F_1
  F_1 moves at the velocity $f(v)$ relative to F_2

A group of clocks $C_i (i = 1 - N)$ located at x_i in F_1.



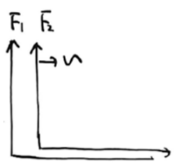
t_0: O' of F_2 passes $x_0 < x_i$
$t_i = t_0 + (x_i - x_0)/v$: O' of F_2 passes x_i



Using the motion of O of F_1 to synchronize clocks in F_2.

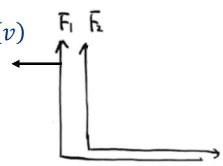
Clock synchronization

- **Corollary 2:** $f(v) = -v$.



Assume F_1 moves at the velocity $f(v)$ relative to F_2.

$f(f(v)) = v$
$f(v) = -f(-v)$ ⇒ $f(v) = \pm v$



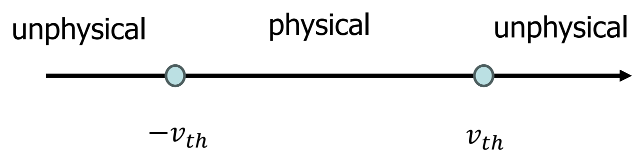
⇒ $f(v) = -v$

- Later we will prove the universal upper limit of speeds v_{th} in all inertial frames.

- Shift speed unit to $v_{th} = c$ → clock synchronization in all frames consistently.

Physical speeds

- **Definition1:** A speed v is physical if a particle can perform a uniform rectilinear motion with such a speed.
- **Lemma1:** If v is physical, then u is physical for the entire range of $0 \le u < v$.
- **Lemma2:** If there exists a threshold v_{th}, v is physical for $0 \le u < v_{th}$, and nonphysical for $u > v_{th}$.



later we will prove v_{th} is frame-independent

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Interface between EM with mechanics

- **Axiom 3:** q is independent on its state of motion and invariant under frame transformations.
- **Definition 2:** Define $\boldsymbol{E}$ and $\boldsymbol{B}$-fields via mechanical observables

$$\boldsymbol{F_E} = q\boldsymbol{E} \qquad \boldsymbol{F_B} = \frac{\boldsymbol{v}}{c_1} \times q\widetilde{\boldsymbol{B}} \qquad \text{at } \boldsymbol{v} \to \boldsymbol{0}.$$

Energy conservation → $\boldsymbol{F_B} \perp \boldsymbol{v}$

c_1 an arbitrary const carrying the speed unit, $\tilde{B}/c_1$ invariant

- **Definition 3:** q is even under time-reversal and spatial inversion operation.

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Linear transformation of EM fields

- **Postulate 2:** $\boldsymbol{F_B}$'s linear dependence on v is valid for all physical speeds.

$$\boldsymbol{F} = q(\boldsymbol{E} + \frac{\boldsymbol{v}}{\boldsymbol{c_1}} \times \tilde{\boldsymbol{B}})$$

- **Corollary 3:** Transformation of E and B between two inertial frames should be linear.

Since force satisfies the rule of superposition, E and B fields also satisfies this rule in all inertial frames. Hence the transformation between inertial frames should be linear.

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Fundamentals of EM

- **Postulate 3:** Maxwell Eqs for steady EM fields established in one frame (named Frame R).

$$\oint\!\!\!\oint \mathbf{E} \cdot \mathbf{dA} = 4\pi Q,$$
$$\oint \mathbf{E} \cdot d\mathbf{l} = 0,$$
$$\oint\!\!\!\oint \tilde{\mathbf{B}} \cdot \mathbf{dA} = 0,$$
$$\oint \tilde{\mathbf{B}} \cdot d\mathbf{l} = \frac{4\pi}{c_2} I,$$

$$c_1 \rightarrow kc_1, c_2 \rightarrow \frac{c_2}{k}, \tilde{B} \rightarrow \tilde{B}k$$
$$c = \sqrt{c_1 c_2}, \quad B = \tilde{B}\sqrt{c_2/c_1}$$

$$\boldsymbol{F_B} = q\frac{\boldsymbol{v}}{c} \times \boldsymbol{B}$$
$$\oint d\boldsymbol{l} \cdot \boldsymbol{B} = \frac{4\pi}{c} I$$

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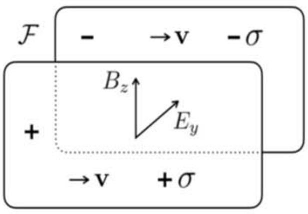
Measurement of *c* without light

in the spirit of Weber-Kohlarusch experiment (1855)

In principle, c defined in Eq. (6) can be measured experimentally via the ratio between the electrostatic force and the magneto-static force. Consider two infinitely large parallel plates with surface electric charge densities $\pm\sigma$ respectively. The attractive force density, *i.e.*, the pressure, on each plate can be calculated via Gauss's law Eq. (2) as $P_E = 2\pi\sigma^2$. Similarly, for two parallel plates carrying electric surface current densities $\pm K$ in opposite directions, respectively. The repulsive force density on each plate can be calculated via Ampère's law Eq. (9) as $P_B = 2\pi K^2/c^2$. Then c is expressed as

$$c = \frac{K}{\sigma}\sqrt{\frac{P_E}{P_B}}. \tag{A2}$$



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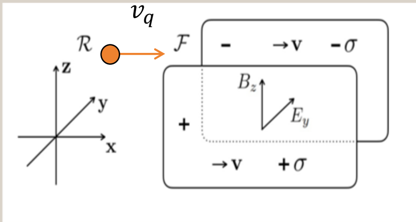
R系中的电荷漂移运动？

- R系中无限大平行板电容器以速度v运动
- 面电荷密度σ，线电流密度$K = \sigma v$



高斯定律 → $E_y = \frac{\sigma}{\epsilon_0}$

安培定律 → $B_z = \mu_0 K$

$$\Rightarrow \boldsymbol{B} = \mu_0\epsilon_0 \boldsymbol{v} \times \boldsymbol{E} = \frac{\boldsymbol{v}}{\boldsymbol{c}^2} \times \boldsymbol{E}$$

- R系中存在物理的电荷漂移速度吗？姑且假设有，则

$$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \quad \Rightarrow \quad v_q = \frac{E_y}{B_z} = c^2/v$$

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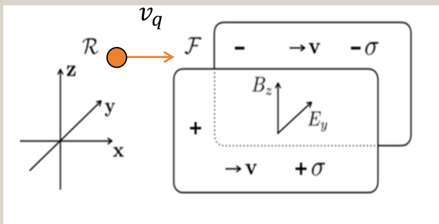
F系：平行板电容器的随动参照系

- F系只有静电荷，没有电流。
- 时间反演对称性 → $B_z' = 0$，$E_y' \neq 0$



$$F' = qE_y' \neq 0$$

- 在F系中，q受外力作用，产生加速度。
- 和力学相对性原理发生矛盾

⟹ R系中的v 和 $v_q = \frac{c^2}{v}$ 不能同时是物理的。

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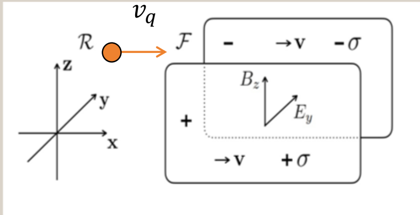
物理速度上界的有限性

- 定理1：R 中的物理速度存在上限 v_{th}

$$0 \le v < v_{th} \le c$$



如果速度上界$v_{th} = +\infty$，则v 和v_q 都是物理的，与相对性原理矛盾。

如果速度上界$v_{th} > c$：令 $v \to c$，则 $v_q = \frac{c^2}{v} \to c \Rightarrow$ 则$v, v_q < v_{th}$，与相对性原理矛盾。

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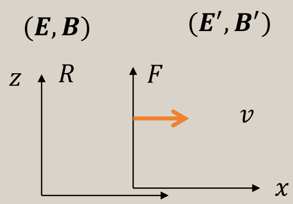
惯性系间的电磁场变换

- 惯性系 R：安培定律和高斯定律成立
- 惯性系 F 相对于 R 以物理速度 v 运动
- 线性变换 + 对称性分析 ⇒



$$E_x \to E_x', \; B_x \to B_x'$$

$$M(v) = \begin{pmatrix} a & b \\ f & d \end{pmatrix}$$

$$E_y' = aE_y + b(cB_z)$$
$$cB_z' = fE_y + d(cB_z)$$

$y \to z,$ $z \to -y$

$$E_z' = aE_z - b(cB_y)$$
$$-cB_y' = fE_z - d(cB_y)$$

a, b, d, f 和E,B 无关, 仅依赖v

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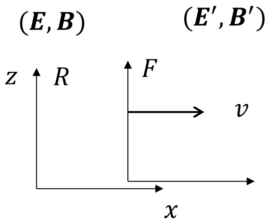
Velocity addition I

- Frame R: Maxwell Eqs for steady fields apply.
- Frame F moves at a physical velocity v relative to R
- Linear transformation + symmetry analysis ⇒



$$E_x \to E_x',\ \ B_x \to B_x' \qquad M(v) = \begin{pmatrix} a & b \\ f & d \end{pmatrix}.$$

$$\begin{pmatrix} E_y' \\ B_z' \end{pmatrix} = \begin{pmatrix} a & b \\ f & d \end{pmatrix} \begin{pmatrix} E_y \\ B_z \end{pmatrix} \qquad \begin{pmatrix} E_z' \\ -B_y' \end{pmatrix} = \begin{pmatrix} a & b \\ f & d \end{pmatrix} \begin{pmatrix} E_z \\ -B_y \end{pmatrix}$$

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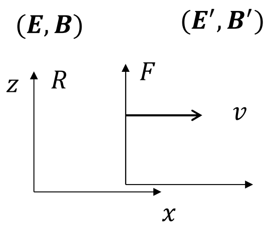
Velocity addition between frames F and R



- Set EM configuration with

$$\frac{E_y}{B_z} = \frac{v}{c} = \beta$$

drift motion: q moves at v in R (rest in F) →

$$E_y' = aE_y + bB_z = 0.$$

$$\frac{b}{a} = -\frac{E_y}{B_z} = -\beta$$

- Set EM configuration with $B_z/E_y = v/c = \beta$

realized by the plate capacitor moving at v in R (rest in F) →

$$B_z' = fE_y + dB_z = 0$$

$$\frac{f}{d} = -\frac{B_z}{E_y} = -\beta$$

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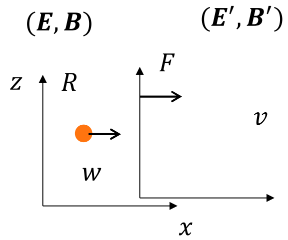
- Charge q of a general velocity $w < v_{th}$ in R
- Set in R: $E_y/B_z = w/c = \beta_w$ → q's drift motion
- In F, q remain a drift motion but its velocity → u

$$\frac{u}{c} = \frac{E'_y}{B'_z} = \frac{aE_y + bB_z}{fE_y + dB_z} = \frac{a}{d}\frac{\beta_w - \beta}{1 - \beta_w \beta}$$

- Set in R: $E_y = 0$, i.e., $w = 0$

→ $u = -v$ → $\frac{a}{d} = 1$



⇒ $$\frac{u}{c} = \frac{\beta_w - \beta}{1 - \beta_w \beta}$$

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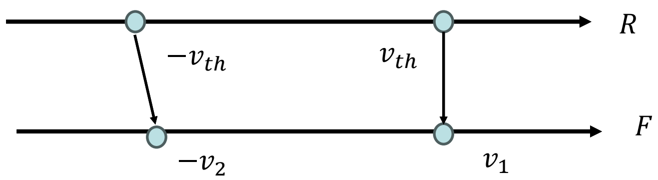
Proof: $v_{th} = c$ and its universality

- Frame F moves at v relative to R $\quad \beta = \frac{v}{c} \quad \beta_{th} = \frac{v_{th}}{c}$
- $\pm v_{th}$ in frame R → v_1, and $-v_2$ in frame F

$$\beta_1 = \frac{v_1}{c} = \frac{\beta_{th} - \beta}{1 - \beta_{th}\beta} \qquad \beta_2 = \frac{v_2}{c} = \frac{\beta_{th} + \beta}{1 + \beta_{th}\beta}$$



- Isotropy in F → $v_1 = v_2$ → $\beta_{th} = 1 = \beta_1 = \beta_2$

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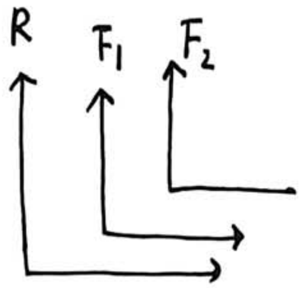
(*) Velocity addition II



F_1 moves at v_{1R} relative to R-frame

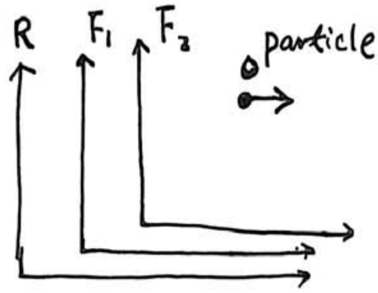
F_2 moves at v_{2R} relative to R-frame

F_2 relative to F_1

$$\beta_{21} = \frac{v_{21}}{c} = \frac{\beta_{2R} - \beta_{1R}}{1 - \beta_{2R}\beta_{1R}} \quad \text{where} \quad \beta_{1R} = \frac{v_{1R}}{c}, \quad \beta_{2R} = \frac{v_{2R}}{c}$$

It's easy to have $\beta_{12} = -\beta_{12}$, then the relative velocities between two frames are opposite to each other.

Now we consider the velocity addition between two general frames.



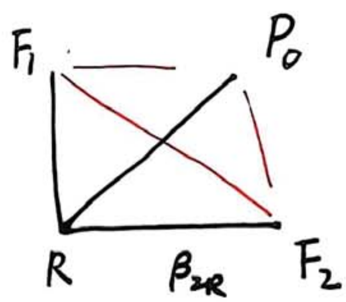
particle moves at $v_{01}\hat{x}$ relative to F_1, F_2 moves at v_{21} to F_1,

Then what's the velocity of the particle relative to F_2 ?

- We introduce R-frame as the intermediate frame.

Assume F_2 moves at $v_{2R}\hat{x}$ relative to R; F_1 moves at $V_{1R}\hat{x}$ relative to R.

$\Delta F_1 F_2 R)$



$$(*)\ \beta_{12} = \frac{\beta_{1R} - \beta_{2R}}{1 - \beta_{1R}\beta_{2R}}$$

- Consider the triangle $\Delta F_1 PR$

we have the velocity , β_{01}, β_{1R} β_{0R}

$$(**)\ \beta_{01} = \frac{\beta_{0R} - \beta_{1R}}{1 - \beta_{0R}\beta_{1R}}$$

- Consider $\Delta F_2 PR$, we have the velocity β_{02}, β_{0R}, β_{2R}

$$(**)\ \beta_{02} = \frac{\beta_{0R} - \beta_{2R}}{1 - \beta_{0R}\beta_{2R}}$$

Then we check the relation among $\Delta P_0 F_1 F_2$

$$\frac{\beta_{01} + \beta_{12}}{1 + \beta_{01}\beta_{12}} = \frac{\frac{\beta_{0R} - \beta_{1R}}{1 - \beta_{0R}\beta_{1R}} + \frac{\beta_{1R} - \beta_{2R}}{1 - \beta_{1R}\beta_{2R}}}{1 + \frac{\beta_{0R} - \beta_{1R}}{1 - \beta_{0R}\beta_{1R}} \frac{\beta_{1R} - \beta_{2R}}{1 - \beta_{1R}\beta_{2R}}}$$

$$= \frac{(\beta_{0R} - \beta_{1R})(1 - \beta_{1R}\beta_{2R}) + (\beta_{1R} - \beta_{2R})(1 - \beta_{0R}\beta_{1R})}{(1 - \beta_{0R}\beta_{1R})(1 - \beta_{1R}\beta_{2R}) + (\beta_{0R} - \beta_{1R})(\beta_{1R} - \beta_{2R})}$$

$$= \frac{(\beta_{0R} - \beta_{2R})(1 - \beta_{1R}^2)}{(1 - \beta_{0R}\beta_{2R})(1 - \beta_{12}^2)} = \frac{\beta_{0R} - \beta_{2R}}{1 - \beta_{0R}\beta_{2R}} = \beta_{02}$$

Hence

$$\boxed{\beta_{02} = \frac{\beta_{01} + \beta_{12}}{1 + \beta_{01}\beta_{12}}}$$