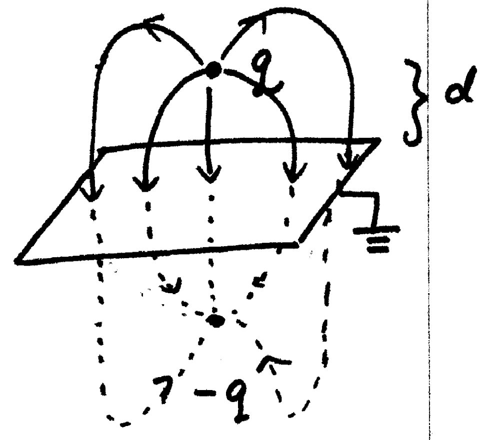
Lect 13 method of images

§: infinitly large grounded conducting plane, with a point charge q above with a distance d.



trick: forget the boundary, but replace with an image charge $-q$ at the symmetric position $-d$. We know that for such a two-charge system, the bi-section plane: the electric potential $V = \frac{q}{|\vec{r}-\vec{r}_q|} - \frac{q}{|\vec{r}-\vec{r}_{-q}|} = 0.$

Thus in the upper half space, we have the charge q, and $\underbrace{\text{the plane}}_{xy}$ with $V=0$. According to the uniqueness theorem, the $\vec{E}$ distribution is just the solution to the original problem! We don't need to solve boundry conditions.

$$V(x,y,z) = \frac{q}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{q}{\sqrt{x^2+y^2+(z+d)^2}}$$

The induced surface charge: the other side is inside meatal $\Rightarrow E=0$

$$\sigma = -\frac{1}{4\pi}\frac{\partial V}{\partial z}\Big|_{z=0} = \frac{-q}{4\pi}\left[\frac{-\frac{1}{2}(z-d)\cdot 2}{(x^2+y^2+(z-d)^2)^{3/2}} - \frac{-\frac{1}{2}(z+d)\cdot 2}{(x^2+y^2+(z+d)^2)^{3/2}}\right]\Bigg|_{z=0}$$

$$= \frac{-qd}{2\pi}\frac{1}{(x^2+y^2+d^2)^{3/2}}$$

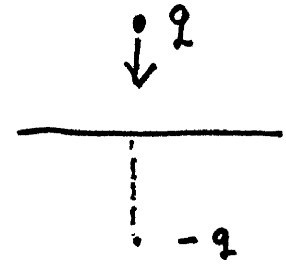
how much charge induced on the plane?

$$Q = \int \sigma \, da = -\frac{qd}{2\pi} \int_0^{+\infty} \frac{1}{(\rho^2+d^2)^{3/2}} \rho \, d\rho \, d\phi = -\frac{qd}{2} \int_0^{+\infty} \frac{dx}{(x+d^2)^{3/2}}$$

$$= -\frac{qd}{2}(-2)(x+d^2)^{-1/2} \Big|_0^{+\infty} = qd\left[0 - [d^2]^{-1/2}\right] = -q.$$ ← overall charge neutral

Force and energy: The force exerted on q is apparently

$$\vec{F} = -\frac{q^2}{(2d)^2}\hat{z}$$



The energy stored in such a system

$$W = +-\int_\infty^d \vec{F}\cdot dl = q^2 \int_\infty^d \frac{1}{(2z)^2}\, dz = \frac{-q^2}{4d}$$

which is the half of the dipole system. The reason is that we only have half of the space, thus the energy stored is also half.

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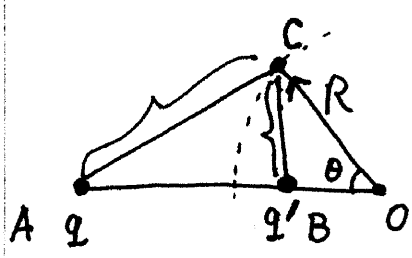

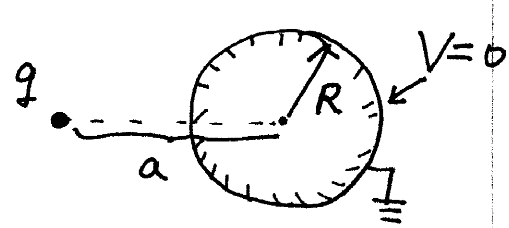

Suppose that q is put at A, We denote a point B on the line segment OA, and satisfy $OB \cdot OA = R^2$. Then for ΔOBC and ΔOCA, we have $\frac{OA}{OC} = \frac{OC}{OB}$ and they share $\angle COB = \angle AOC$.

$\Rightarrow \Delta OBC \backsim \Delta OCA. \Rightarrow \frac{BC}{AC} = \frac{OC}{OA} = \frac{R}{a}.$ This is a fixed value regardless of the location of C.

~~if we assign a charge q' and B, with~~

$$\frac{q'}{q} = -\frac{BC}{AC} = -\frac{R}{a}.$$

$$\Rightarrow \frac{q'}{|BC|} + \frac{q}{|AC|} = 0$$ then the potential on the sphere is zero.

The location of the image at $\boxed{OB = \frac{R^2}{a}, \text{ and } q' = -\frac{R}{a} q}$.

$$V(r) = \frac{q}{|\vec{r} - \vec{r}_A|} + \frac{q'}{|\vec{r} - \vec{r}_B|}.$$

The force on A is along the direction of OA toward O.

$$|F| = \frac{qq'}{|AB|^2} = \frac{q^2 R a}{(a^2 - R^2)^2}$$