Lect 19 Linear Dielectrics

For many material, at small fields, the polarization is proportional to $\vec{E}$ described as $\vec{P} = \chi \vec{E}$, where χ is called electric susceptibility of the medium. This is called linear dielectrics. $\vec{E}$ here is the total field, not the external field $\vec{E}_0$. So the entire process is a self-consistent process.

$$\boxed{\vec{E}_{tot} = \vec{E}_0 + \vec{E}_{ind}} \xrightarrow{\text{linear dielec}} \boxed{\vec{P} = \chi \vec{E}_{tot}}$$

$$\rho_b = -\nabla \cdot \vec{P}$$

Polarization generate induced field.

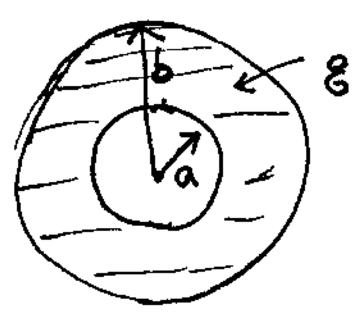
we have $\vec{D} = \vec{E}_{tot} + 4\pi \vec{P} = (1 + 4\pi\chi)\vec{E}_{tot}$

dielectric constant $\boxed{\varepsilon = 1 + 4\pi\chi}$.

For a homogeneous medium, the electric field generated by a free charge

$$\vec{E} = \frac{Q}{\varepsilon r^2}\hat{r}.$$

Example: a metal sphere with radius "a" carries a ~~total~~ charge Q, it's surrounded out to radius b, by linear dielectric material of dielectric const ε.



Solution: The only free charge is on the sphere "a". $\Rightarrow$ $\vec{D} = \begin{cases} \dfrac{Q}{r^2}\hat{r} & \text{for } r > a \\ 0 & \text{for } r < a \end{cases}$

The dielectric const is $\varepsilon = \begin{cases} 1 & r > b \\ \varepsilon & a < r < b \end{cases}$

$$\Rightarrow \vec{E}(\vec{r}) = \begin{cases} \frac{Q\,\hat{r}}{r^2} & \text{for } r > b \\ \frac{Q\,\hat{r}}{\varepsilon r^2} & \text{for } b > r > a \\ 0 & \text{for } r < a \end{cases}$$

$\Rightarrow$ the electric potential

$$V(r) = -\int_\infty^r dr\, E(r)$$

$$V(r) = \begin{cases} \frac{Q}{r} & \text{for } r > b \\ \frac{Q}{b} + \left[\frac{Q}{r} - \frac{Q}{b}\right]\frac{1}{\varepsilon} & \text{for } b > r > a \\ \frac{Q}{b} + \left[\frac{Q}{a} - \frac{Q}{b}\right]\frac{1}{\varepsilon} & \text{for } r < a \end{cases}.$$

The polarization $\vec{P} = \chi \vec{E} = \frac{\chi}{\varepsilon}\frac{Q}{r^2}\hat{r}$ for $a < r < b$

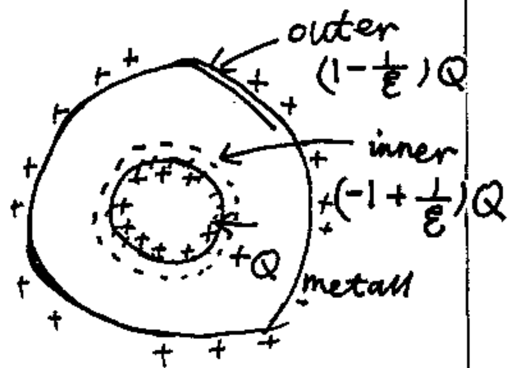
$$\Rightarrow \rho_b = -\nabla\cdot\vec{P} = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 P\right) = 0.$$

$$\sigma_b = \vec{P}\cdot\hat{n} = \begin{cases} \frac{\chi}{\varepsilon}\frac{Q}{b^2} & \text{at } r = b \\ -\frac{\chi}{\varepsilon}\frac{Q}{a^2} & \text{at } r = a. \end{cases}$$

The ~~total~~ bound charge at the sphere of $r = a$ is $-\frac{4\pi\chi}{\varepsilon}Q = -\left(\frac{\varepsilon - 1}{\varepsilon}\right)Q = \left(-1 + \frac{1}{\varepsilon}\right)Q$

at the outer sphere $r = b$ is $\left(1 - \frac{1}{\varepsilon}\right)Q$.

Clearly in the region $a < r < b$, the $+Q$ charge of the metal is partially screened to $\frac{1}{\varepsilon}$. As $\varepsilon \to +\infty$, we recover metall.



§ Boundary value problem with linear dielectrics

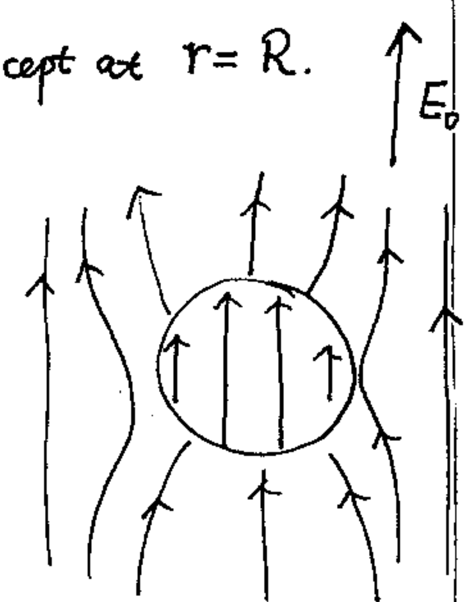
ex: a sphere of homogeneous linear dielectric material with ε is placed in an otherwise uniform electric field $\vec{E}_0$. Find the electric field distribution.

There's no free charge $\rho_f = 0$. we know $\rho_b = -\nabla\cdot\vec{P} = -\nabla[\frac{4\pi\chi}{\varepsilon}\vec{D}]$

$$= -\frac{4\pi\chi}{\varepsilon}\rho_f = 0$$

$\Rightarrow$ $\rho_{tot} = 0$ at the place of $\rho_f = 0$

$\Rightarrow$ we can still use $-\nabla^2 V = 0$ everywhere except at $r = R$.



again due to the axial symmetry, we have

$V_{in}(r, \theta)$ for $r \le R$, $V_{out}(r, \theta)$ for $r > R$.

i) $V_{in} = V_{out}$ at $r = R$.

ii) the norm component of Displacement is continuous

$$\varepsilon \frac{\partial V_{in}}{\partial r} = \frac{\partial V_{out}}{\partial r} \quad \text{at } r = R$$

iii) at $r \to \infty$ $V_{out} = -E_0 r \cos\theta$.

} boundary conditions

According to the general solution

$$V_{in}(r, \theta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta)$$

$$V_{out}(r, \theta) = -E_0 r \cos\theta + \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta)$$

i) $\Rightarrow \displaystyle\sum_{\ell=0}^{\infty} A_\ell R^\ell P_\ell(\cos\theta) = -E_0 R\cos\theta + \sum_{\ell=0}^{\infty} \frac{B_\ell}{R^{\ell+1}} P_\ell(\cos\theta)$

$$\begin{cases} A_\ell R^\ell = \dfrac{B_\ell}{R^{\ell+1}} & \text{for } \ell \neq 1 \\ A_1 R^\ell = -E_0 R + \dfrac{B_1}{R} & \text{for } \ell = 1 \end{cases}$$

ii) $\Rightarrow \displaystyle \varepsilon \sum_{\ell=0}^{\infty} \ell A_\ell R^{\ell-1} P_\ell(\cos\theta) = -E_0\cos\theta + \sum_{\ell=0}^{\infty} -(\ell+1)\frac{B_\ell}{R^{\ell+2}} P_\ell(\cos\theta)$

$$\Rightarrow \begin{cases} \varepsilon \ell A_r R^{\ell-1} = -\dfrac{\ell+1}{R^{\ell+2}} B_\ell & \text{for } \ell \neq 1 \\ \varepsilon A_1 = -E_0 - \dfrac{2B_1}{R^3} & \text{for } \ell = 1 \end{cases}$$

$\Rightarrow \quad A_\ell = B_\ell = 0 \quad$ for $\ell \neq 1$

and $A_1 = \dfrac{-3}{\varepsilon+2} E_0$ and $B_1 = \dfrac{\varepsilon-1}{\varepsilon+2} R^3 E_0$

$\Rightarrow \quad V_{in} = -\dfrac{3E_0}{\varepsilon+2} r\cos\theta = -\dfrac{3E_0}{\varepsilon+2} z \Rightarrow \boxed{\vec{E}_{in} = \dfrac{3}{\varepsilon_r+2}\vec{E}_0 < \vec{E}_0}$

$V_{out} = -E_0 r\cos\theta + \dfrac{\varepsilon-1}{\varepsilon+2} R^3 E_0 \dfrac{\cos\theta}{r^2}$ $\qquad \vec{\mathbb{P}}\cdot Vol$

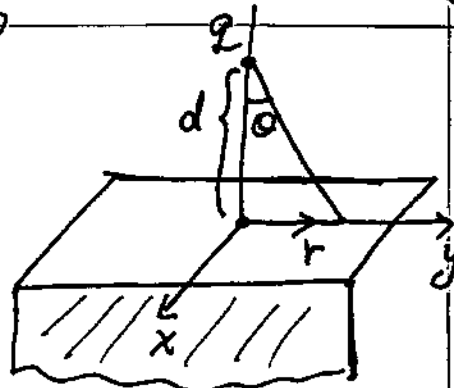
$\boxed{V_{out} = -E_0 r\cos\theta + \dfrac{\vec{p}\cdot\hat{z}}{r^2}}$ where $\vec{p} = \underbrace{\dfrac{\varepsilon-1}{\varepsilon+2} E_0 \dfrac{3}{4\pi}} \cdot \left(\dfrac{4\pi}{3}R^3\right) =$

The polarization inside $\boxed{\vec{\mathbb{P}} = \chi \vec{E}_{in} = \dfrac{(\varepsilon-1)}{4\pi}\vec{E}_{in} = \left(\dfrac{\varepsilon-1}{\varepsilon+2}\dfrac{3}{4\pi}\right)\vec{E}_0}$

$\Rightarrow$ The potential outside is a superposition of the background field + a dipole field with the ~~magnet~~ polarization $\left(\dfrac{\varepsilon-1}{\varepsilon+2}\dfrac{3}{4\pi}\right)\vec{E}_0$.

The induced suface $\boxed{\sigma_b(\theta) = \vec{\mathbb{P}}\cdot\hat{n} = \dfrac{\varepsilon-1}{\varepsilon+2}\dfrac{3}{4\pi} E_0 \cos\theta}$

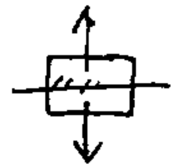
Example : Suppose that the entire reigen below $z=0$ in filled by a material with dielectric constant ε.



Surface bound $\underbrace{\text{charges must}}_{\sigma_b(r)}$ appear on the plane of $z=0$. There should be no induced body charge $\underbrace{\rho_b(r)}_{\text{at } z<0.}$, because $\rho_f = 0$. at $z<0$.

$$\sigma_b(r) = \vec{P}\cdot\hat{n} = \frac{\varepsilon-1}{4\pi} E_z(0^-) \leftarrow \text{just below } z=0.$$

$E_z(0^-)$ is a super-position of the field generated by q and by the surface charge $\sigma_b(r)$.



The first contribution $\dfrac{q}{(d^2+r^2)} \dfrac{-d}{(d^2+r^2)^{1/2}}$

the second one is releated by $\sigma_b(r)$ itself $-2\pi\sigma_b(r)$.

$$\Rightarrow \quad \sigma_b(r) = \frac{\varepsilon-1}{4\pi} E_z(0^-) = \frac{\varepsilon-1}{4\pi}\left[\frac{-qd}{(d^2+r^2)^{3/2}} - 2\pi\sigma_b(r)\right]$$

$$\text{or } \left[1+\frac{\varepsilon-1}{2}\right]\sigma_b(r) = \frac{\varepsilon-1}{4\pi}\frac{-qd}{(d^2+r^2)^{3/2}} \Rightarrow \boxed{\sigma_b(r) = \frac{\varepsilon-1}{\varepsilon+1}\frac{1}{2\pi}\frac{-qd}{(d^2+r^2)^{3/2}}}$$

The total induced charge $2\pi\int r\,dr\,\sigma_b(r) = -\left(\dfrac{\varepsilon-1}{\varepsilon+1}\right) q$.

The total electric field generated by $\sigma_b(r)$ toward q is

$$E_z = \int d\theta\, r\,dr \frac{\sigma_b(r)}{(d^2+r^2)} \frac{(+d)}{(d^2+r^2)^{1/2}}$$

$$E_z = \pi \int_0^{+\infty} dr^2 \left(\frac{\varepsilon-1}{\varepsilon+1}\right) \frac{-q}{2\pi} \frac{d}{(d^2+r^2)^3} = -\frac{1}{2}\frac{\varepsilon-1}{\varepsilon+1}\frac{q}{d^2}\int_0^{+\infty} d\left(\frac{r^2}{d^2}\right)\frac{1}{(1+r^2/d^2)^3}$$

$$= -\frac{1}{2}\frac{\varepsilon-1}{\varepsilon+1}\frac{q}{d^2}\int_0^{\infty} dx \frac{1}{(1+x)^3} \qquad \int\frac{dx}{(1+x)^3} = -\frac{1}{2}\frac{1}{(1+x)^2}$$

$$= -\frac{1}{4}\frac{\varepsilon-1}{\varepsilon+1}\frac{q}{d^2}$$

$\Rightarrow$ $\vec{F}_{tot}$ is along the z-axis $\Rightarrow$ $\vec{F}_{tot} = -\frac{1}{4}\frac{\varepsilon-1}{\varepsilon+1}\frac{q^2}{d^2}\hat{z}$

if we set $\varepsilon = \infty$, we have

$\vec{F}_{tot} = -\frac{1}{4}\frac{q^2}{d^2}\hat{z}$ as in the case of metallic plane.

§ Energy in dielectric systems.

The energy ~~not~~ dose only stored in the field, but also in the material.

Suppose we do a little work to increase the free charge, certainly the polarization and the bound charge also change. So during this process not only work is done by us to free charge, but also work is done to the bound charge by the medium. The second part will be added into the field energy as ~~the~~ part of the total energy stored in the medium.

Only the free charge that we can controll. — the right amount energy ~~that~~ we can use.

$$\text{Work} = \int (\Delta\rho_f)\, V\, d\tau = \frac{1}{4\pi}\int \Delta(\nabla\cdot\vec{D})\, V\, d\tau$$

$$\text{Work} = \frac{1}{4\pi}\int \nabla\cdot(\Delta\vec{D})\, V\, d\tau = \frac{1}{4\pi}\left[\int \nabla\cdot(\Delta\vec{D}\, V)\, d\tau - \int \Delta\vec{D}\cdot\nabla V\, d\tau\right]$$

$$= \frac{1}{4\pi}\oint \Delta\vec{D}\, V\, da + \frac{1}{4\pi}\int \Delta\vec{D}\cdot\vec{E}\, d\tau$$

set ~~suf~~ surface to infinity

$$\Rightarrow \quad dW = \frac{1}{4\pi}\int d\vec{D}\cdot\vec{E}\, d\tau$$

for a linear media $\Rightarrow$ $D = \varepsilon\vec{E}$ $\Rightarrow$ $d\vec{D}\cdot\vec{E} = d\vec{E}\cdot\vec{D}$

$$\Rightarrow \quad dW = \frac{1}{8\pi}d\int \vec{D}\cdot\vec{E}\, d\tau \quad \Rightarrow \quad \boxed{W = \frac{1}{8\pi}\int \vec{D}\cdot\vec{E}\, d\tau}$$

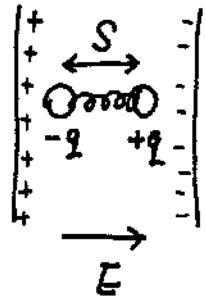
example: the energy change during polarization

① we know that in order to charge capacitor C, to voltage V, the energy stored $\quad dW = V'dq = CV'dV' \Rightarrow W = C\int_0^V V'dV' = \frac{1}{2}CV^2$.

-If the medium between the plates is just vacum, we know the energy density is simply $\frac{1}{8\pi}E^2 = \frac{1}{8\pi}\left(\frac{V}{d}\right)^2$

the volum is $A\cdot d$ $\Rightarrow$ $\underbrace{\frac{1}{8\pi}E^2\, A\cdot d}_{\text{~~total~~ energy is}} = \frac{1}{2}\,\frac{A}{4\pi d}\,V^2$, which is just $\frac{1}{2}CV^2$.

If we use the medium of dielectric constant ε, then "C" ~~is~~ increases to $\frac{\varepsilon A}{4\pi d}$. If the voltage is the same V, and so does the E, the energy stored in the medium increase to $\frac{1}{8\pi}\int \varepsilon E^2 d\tau$. But how does the extra amount of energy is stored?



Let us just consider a polarizable molecule, two charges $\pm q$, connected by a spring. Its dipole $p = qS$. Suppose that at the field strength E, the spring is elongated at ds, the E-field does work to the molequle.

$$\boxed{dw' = E\,q\,ds = \vec{E}\cdot d\vec{p}}$$ ← this work is stored as energies, which contains two parts ① the elongated spring ② the field of the diapoles.

In parctice, we do not make this distinction, because both belong to energy of molecule structure. The point is that the battary does the amount of work on molecules to change their structure.

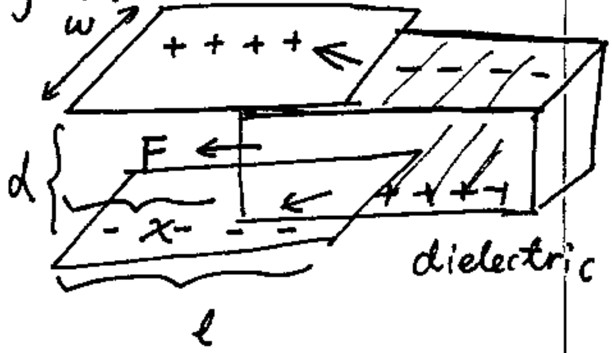
This ~~amm~~ amount of work is just

$$dw' = \vec{E}\cdot d\frac{(\vec{D}-\vec{E})}{4\pi} = \frac{1}{4\pi}\vec{E}\cdot d\vec{D} - \frac{\vec{E}}{4\pi}\cdot d\vec{E}$$

$$\Rightarrow \boxed{dw = dw' + \underbrace{\frac{\vec{E}\,d\vec{E}}{4\pi}}_{\text{the energy change from field.}} = \frac{1}{4\pi}\vec{E}\cdot d\vec{D}}$$

(9)

§ forces: dielectric is attracted into the field:



Let us calculate the energy of the system as a function of x: $W(x)$.

If I pull out the dielectric a little $x+\Delta x$, the energy changes to $W(x+\Delta x)$, thus

$dW = F\,dx$, thus the force acting on the dielectric due to the charges on the plates is $-\frac{dW}{dx}$.

$$C = \frac{\omega x}{4\pi d} + \frac{\omega(l-x)}{4\pi d}\epsilon,$$

$$W = \frac{1}{2}CV^2 = \frac{1}{2}\frac{(CV)^2}{C} = \frac{1}{2}\frac{Q^2}{C(x)}$$

$$F = -\frac{dW}{dx} = \frac{1}{2}\frac{Q^2}{C^2}\frac{dC}{dx} = \frac{1}{2}V^2\frac{dC}{dx} = \frac{1}{2}V^2\frac{\omega}{4\pi d}(1-\epsilon)$$

$$= -\frac{1}{2}V^2\frac{\omega}{4\pi d}(\epsilon-1)$$

We need to use fixing Q, rather than fixing V, Because the capacitor is open. It's voltage is not fixed during the process.