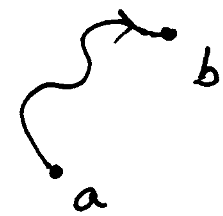
Lect 4 fundment theorems of vecter calculus

§: $\boxed{\text{fundment theorems : integral of a total derivative is determined by the boundary}}$

① Calculus $\displaystyle\int_a^b \frac{df}{dx}\,dx = f(b) - f(a)$ — 1D



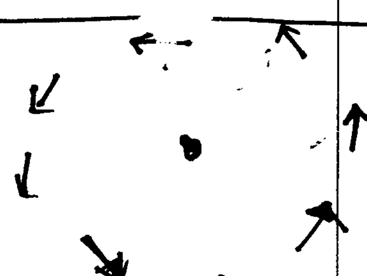
② curved line $dT = \nabla T \cdot d\vec{r}$

$$\underbrace{\int_{\vec{a}}^{\vec{b}} (\nabla T)\cdot d\vec{r}}_{\text{line}} = \int_{T(\vec{a})}^{T(\vec{b})} dT = \underbrace{T(\vec{b}) - T(\vec{a})}_{\text{boundary: two ends}}$$

$\int_a^b \nabla T \cdot d\vec{r}$ is independent of the path from $a \to b$

$\boxed{\oint_a^b \nabla T\, d\vec{r} = 0}$ — Caveat : only valid for single valued function T.

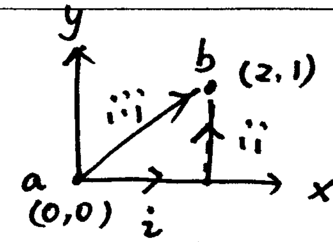
☆ if T is a multi-valued function, say, $\theta(x,y)$ the azimuthal angle of the point (x,y), because θ is only uniquely defined up to $2n\pi$, $\Rightarrow$ $\boxed{\oint \nabla\theta\, d\vec{r} = 2n\pi}$



$\vec{\nabla\theta}\,(x,y)$

in this class, unless explicitly mentioned, we only consider single-valued function

Ex: $T = x y^2$



$$\nabla T = y^2 \hat{x} + 2xy\,\hat{y}$$

$$d\vec{r} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$$

following i + ii $\Rightarrow$ $\int_i \nabla T\, d\vec{r} + \int_{ii} \nabla T\, d\vec{r}$

$$= \int_i y^2 dx + \int_{ii} 2xy\, dy = 0 + 4\int_0^1 y\, dy = 2$$

following iii $y = \frac{x}{2}$ $\Rightarrow$ $\int_{iii} \nabla T \cdot d\vec{r} = \int_{iii} y^2 dx + 2xy\, dy$

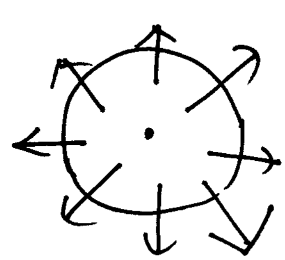
$$= \int_0^2 \frac{x^2}{4} dx + \frac{x^2}{2} dx = \left(\frac{1}{12} x^3 + \frac{x^3}{3}\right)\Big|_0^2 = \frac{8}{4} = 2$$

for both cases $\boxed{\int \nabla T \cdot d\vec{r} = T(2,1) - T(0,0) = 2}$

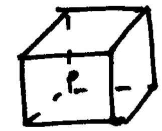
§ fundamental theorem of divergence — Gauss's theorem

$$\oint_V (\nabla \cdot \vec{v})\, d\tau = \oint_S \vec{v} \cdot d\vec{a}$$

$\int$ faucets within the volume = $\oint$ flow out through the surface. boundary



explaination: let's conside a small cube with center (x, y, z) and edge length Δa



then the flux pass the surface:

up & down $\left[v_z(x, y, z+\frac{a}{2}) - v_z(x, y, z-\frac{a}{2}) \right] (\Delta a)^2$

$$= \partial_z v_z \, (\Delta a)^3$$

left & right + front & back $= \left[v_x(x+\frac{a}{2}, y, z) - v_x(x-\frac{a}{2}, y, z) \right] (\Delta a)^2$

$$+ \left[v_y(x, y+\frac{a}{2}, z) - v_y(x, y-\frac{a}{2}, z) \right] (\Delta a)^2$$

$$= (\partial_x v_x)(\Delta a)^3 + (\partial_y v_y)(\Delta a)^3$$

$$\Rightarrow \oint \vec{v} \cdot d\vec{a} = \nabla \cdot \vec{v} \, (\Delta a)^3 = \int_V (\nabla \cdot \vec{v}) \, d\tau .$$

for large volume, you can cut the system into ~~small cubes,~~ a collection of, apply the above result, and add them together.

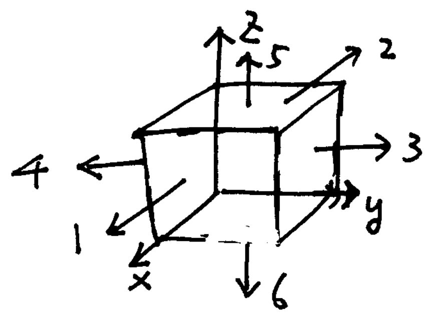
$$\underbrace{\sum \oint \vec{v} \cdot d\vec{a}}_{\text{sum of}} = \oint \vec{v} \cdot d\vec{a}$$

the external surface, all the contribution on internal surfaces cancel

Ex: $\vec{v} = y^2 \hat{x} + (2xy + z^2)\hat{y} + (2yz)\hat{z}$

and the cube.

Check Gauss's law.



$\nabla \cdot \vec{v} = \quad 2x + 2y$

$$\int \nabla\cdot\vec{v} = \int dx\,dy\,dz\; 2(x+y) = 2\int_0^1 dz \int_0^1 dx\,dy\,(x+y) = 4\int_0^1 dz\int_0^1 dy\int_0^1 x\,dx = 2$$

- flux $\int_1 + \cdots \int_6 = \oint d\vec{v}$

$$\int_1 + \int_2 = \int_0^1 dy\,dz\; \underset{x=1}{y^2} - \int_0^1 dy\,dz\; \underset{x=0}{y^2} = 0$$

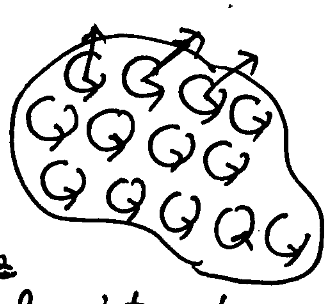
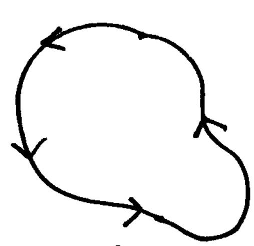
$$\int_3 + \int_4 = \int_0^1 dx\,dz\; \underset{y=1}{(2xy+z^2)} - \int_0^1 dy\,dz\; \underset{y=0}{(2xy+z^2)} = 2\int_0^1 dx\,dz\; x = 1$$

$$\int_5 + \int_6 = \int_0^1 dx\,dy\; \underset{z=1}{(2yz)} - \int_0^1 dx\,dy\; \underset{z=0}{(2yz)} = \int_0^1 dx\,dy\; 2y = 1$$

$$\Rightarrow \int_1 + \cdots \int_6 = 2$$

§ fundamental laws of curls — stoke's theorem

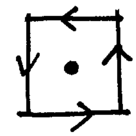
$$\underset{\text{surface}}{\int_S} (\nabla\times\vec{v})\cdot d\vec{a} = \underset{\text{bounday}}{\underline{\oint \vec{v}\cdot d\vec{l}}}$$

 = 

~~one~~ surface integral of $\nabla\times\vec{v}$ | line integral of $\vec{v}$ on the boundary check for a planer version.

$$\int_S \left(\frac{\partial}{\partial x} v_y - \frac{\partial}{\partial y} v_x\right) dx\, dy = \oint \vec{v}\cdot d\vec{l}$$

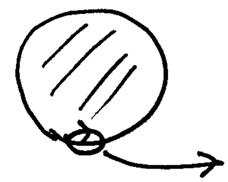
$$\oint \vec{v}\cdot d\vec{l} = \left\{-v_y(x-\tfrac{\Delta a}{2}, y) + v_y(x+\tfrac{\Delta a}{2}, y) + v_x(x, y-\tfrac{\Delta a}{2}) - v_x(x, y+\tfrac{\Delta a}{2})\right\} \Delta a$$



$$= \left(\frac{\partial}{\partial x} v_y - \frac{\partial}{\partial y} v_x\right)(\Delta a)^2 .$$

$\rightarrow$ $\int (\nabla\times\vec{v})\cdot d\vec{a}$ only depend on boundy, but not the surface. • A closed curve in 3D, doesn't uniquly determine a surface. All the surfaces share the same boundary yield the same result.

for a closed surface $\quad \oint (\nabla\times\vec{v})\cdot d\vec{a} = \oint \vec{v}\cdot d\vec{l} = 0$

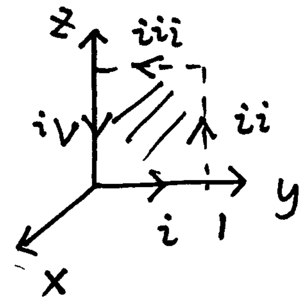


~~contract~~ the boundary curve to a point

$$\boxed{\oint (\nabla\times\vec{v})\cdot d\vec{a} = \int \nabla\cdot(\nabla\times\vec{v})\, d\tau = 0}$$

Ex $\vec{v} = (2xz + 3y^2)\,\hat{y} + 4yz^2\,\hat{z}$,

square surface



$$\nabla\times\vec{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{x} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{y}$$

$$+\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{z} = (4z^2 - 2x)\,\hat{x} + 2z\,\hat{z}$$

$$\int \nabla\times\vec{v}\; d\vec{a} = \int_{x=0} (4z^2 - 2x)\, dy\, dz = 4\int_0^1 dy \int_0^1 z^2 dz = \frac{4}{3}$$

$$\oint \vec{v}\cdot d\vec{l} = \int_i + \cdots \int_{iv} \Rightarrow \int_i + \int_{iii} = \int_i dy\,\underbrace{(2xz+3y^2)}_{x=z=0}$$

$$- \int_{iii} dy\,\underbrace{(2xz+3y^2)}_{x=z=0}$$

$$= \int_0^1 dy\,(3y^2 - 3y^2) = 0$$

$$\int_{ii} + \int_{iv} = \int_0^1 dz\,\underbrace{4yz^2}_{x=0,\,y=1} - \int_0^1 dz\,\underbrace{4yz^2}_{x=0,\,y=0} = 4\int_0^1 dz\, z^2 = \frac{4}{3}$$

$$\Rightarrow \oint \vec{v}\cdot d\vec{l} = 4/3 = \oint \nabla\times\vec{v}$$

§ integral by parts

$$\nabla \cdot (f\vec{A}) = f(\nabla\cdot\vec{A}) + \vec{A}\cdot(\nabla f)$$

$$\Rightarrow \int_V f(\nabla\cdot\vec{A})\,dv = \int_V \left[\nabla\cdot(f\vec{A}) - \vec{A}\cdot(\nabla f)\right]dv$$

$$= \oint f\vec{A}\cdot d\vec{a} - \int_V \vec{A}\cdot\nabla f\,dv$$

in some situations, we can take the surface to infinity.

- if $f\cdot\vec{A}$ decays very quickly, the surface intergral often vanishes!