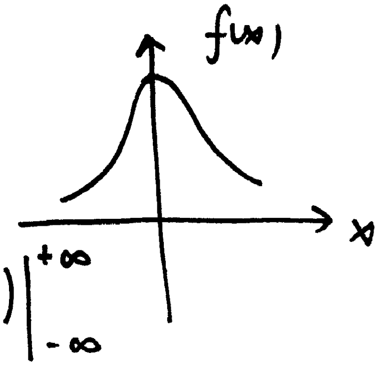
Lect 6 Delta function & other

· §1 define δ-function as a limit of distribution function



$$f(x) = \frac{a}{x^2+a^2} \cdot \frac{1}{\pi}$$

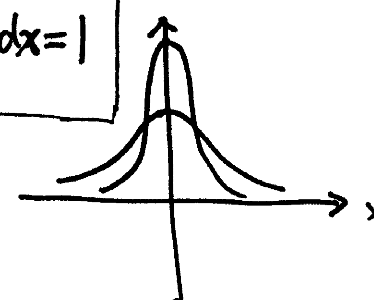
$$\int_{-\infty}^{+\infty} dx\, f(x) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{dx}{a} \frac{1}{\left(\frac{x}{a}\right)^2+1} = \frac{1}{\pi} \operatorname{arctg}\left(\frac{x}{a}\right)\Big|_{-\infty}^{+\infty}$$

$$= 1$$

take the limit of $a \to 0$, then $\int_{-\infty}^{+\infty} dx\, f(x) = 1$ doesn't change.

$$\boxed{\int_{-\infty}^{+\infty} \delta(x)\,dx = 1}$$

but $\lim_{a\to 0} f(x) = \begin{cases} 0 & x \neq 0 \\ +\infty & x = 0 . \end{cases}$



we define $\delta(x) = \lim_{a\to 0} \frac{1}{\pi} \frac{a}{x^2+a^2}$.

There are many other way to define $\delta(x)$ through distribution functi

§2. $$\frac{1}{x \pm i\delta} = P\left(\frac{1}{x}\right) \mp i\pi\delta(x)$$

$$\frac{1}{x \pm ia} = \frac{x \mp ia}{x^2+a^2} = \frac{x}{x^2+a^2} \mp i \frac{a}{x^2+a^2}$$

$$\lim_{a\to 0} \frac{1}{x \pm ia} = P\left(\frac{1}{x}\right) \mp i\pi\delta(x)$$

Principle value

* Properties

$$f(x)\,\delta(x) = f(0)\,\delta(x)$$

$$\int_{-\infty}^{+\infty} f(x)\,\delta(x)\,dx = f(0)\int_{-\infty}^{+\infty}\delta(x)\,dx = f(0)$$

$$\int_{-\infty}^{+\infty} f(x)\,\delta(x-a)\,dx = f(a)$$

$$\delta(f(x)) = \sum_a \frac{\delta(x-x_a)}{|f'(x)|_{x=x_a}}$$ where x_a is the zero of $f(x)$.

Proof: around each zero x_a, $f(x) \approx f'(x_a)(x-x_a)$

for any function $g(x)$ $\displaystyle\int_{a-\epsilon}^{a+\epsilon} dx\,\delta(f(x))\,g(x) = \int_{a-\epsilon}^{a+\epsilon} dx\,\delta[f'(x_a)(x-x_a)]\,g(x)$

$$= \int_{-|f'(x_a)\epsilon|}^{|f'(x_a)\epsilon|} \frac{dx}{|f'(x_a)|}\,\delta(x)\,g\left(\frac{x}{|f'(x_a)|}+x_a\right) = g(x_a)/|f'(x_a)|$$

Sum over all the zeros $\Rightarrow$ $\displaystyle\int_{-\infty}^{+\infty} dx\,\delta(f(x))\,g(x) = \sum_a \frac{g(x_a)}{|f'(x_a)|}$

§ 3D - δ functions

$$\begin{cases} \delta^{(3)}(\vec{r}) = \delta(x)\,\delta(y)\,\delta(z) \\ \int f(\vec{r})\,\delta^{3}(\vec{r})\,d\vec{r} = f(0) \end{cases}$$

example: $\nabla\cdot\left(\frac{\hat{r}}{r^2}\right) = ?$

for any $\vec{r} \neq 0$, $\quad \nabla\cdot\left(\frac{\hat{r}}{r^2}\right) = \partial_x\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}\right) + (x\to y) + (x\to z)$

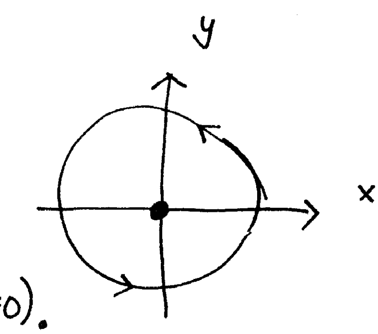
$$= -\frac{3}{2}\,\frac{2(x^2+y^2+z^2)}{r^{5/2}} + \frac{3}{r^{3/2}} = \frac{-3+3}{r^{3/2}} = 0$$

however. $\oint_R \frac{\hat{r}}{r^2}\, d\vec{S} = \frac{4\pi R^2}{R^2} = 4\pi$, which is independent on R.

⇒ this will be consistent if we set $\nabla\cdot\frac{\hat{r}}{r^2} = 4\pi\delta(\vec{r})$

and ⇒ $\nabla^2\left(\frac{1}{r}\right) = -4\pi$

• example $\quad \nabla\times(\nabla\theta(x,y)) = ?$



we know that at any point excep $r=0$,

$\theta(x,y)$ is regular $\Rightarrow \nabla\times\nabla\theta = 0 \quad (r\neq 0)$.

$$\int \nabla\times\nabla\theta(x,y)\, dl = \oint \nabla\theta(x,y)\, dl =$$

area of the circle

$$\theta(x,y) = \mathrm{arctg}\,\frac{y}{x} \Rightarrow \nabla\theta(x,y) = \frac{-\frac{y}{x^2}}{1+\left(\frac{y}{x}\right)^2}\,\hat{x} + \frac{\frac{1}{x}}{1+\left(\frac{y}{x}\right)^2}\,\hat{y}$$

$$= \frac{-y\hat{x} + x\hat{y}}{x^2+y^2} = \frac{1}{r}\,\hat{e}_\varphi$$

$$\Rightarrow \oint \nabla\theta(x,y)\, dl = 2\pi r\,\frac{1}{r} = 2\pi$$

⇒ this will be consistent with $\nabla\times(\nabla\theta(x,y)) = 2\pi\,\delta^{(2)}(\vec{r})$