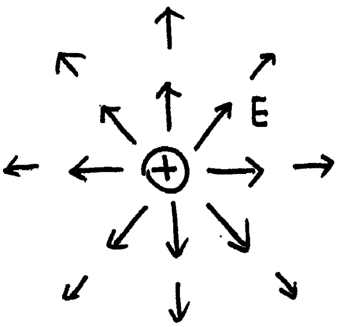
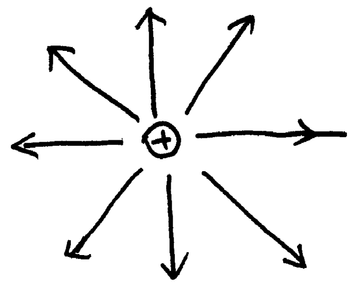
Lect 8 Divergence and curl of electro-static field

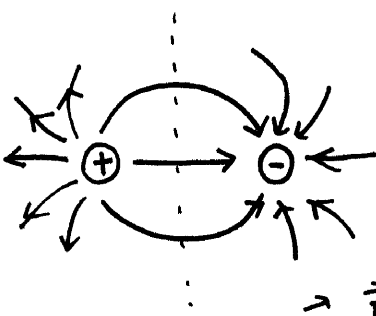
§ field lines : direction represents the direction of field

density represents the field ~~strength~~.



connet the arrows





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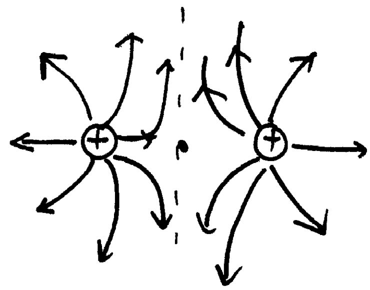
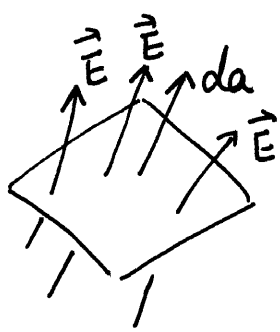


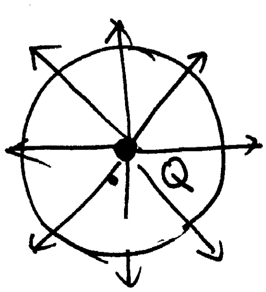
flux



$$\phi_E = \int_S \vec{E} \cdot d\vec{a}$$

how about the flux through a closed surface ?

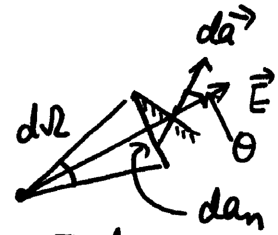
for a point static charge, if the surface is a sphere

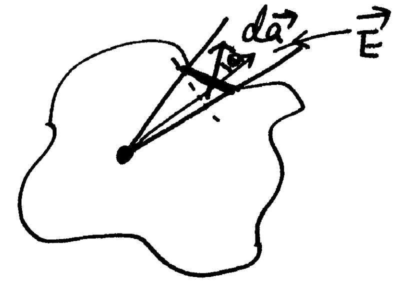


$$\Phi = 4\pi R^2 \cdot \frac{Q}{R^2} = 4\pi Q ,$$ which is independent of radius R.

Actually, it does not matter the concrete shape of the surface.

imagne an area segment



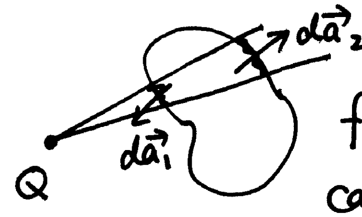


$$d\phi = \vec{E}\cdot d\vec{a} = E\, da \cos\theta = E\, da_n$$

$$da_n = R^2 d\Omega \;, \quad E = \frac{Q}{R^2}$$

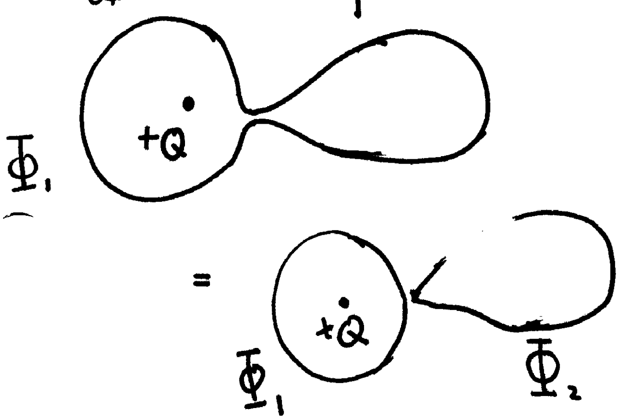
$$\Rightarrow d\phi = Q\, d\Omega \quad \Rightarrow \Phi = \int d\phi = Q \int d\Omega = 4\pi Q.$$

Exer: If Q is outside the surface, then the flux is zero.



the contribution from $\vec{E}\cdot d\vec{a}_1$ and $\vec{E}\cdot d\vec{a}_2$ cancels. $\Rightarrow \Phi = 0.$

another proof



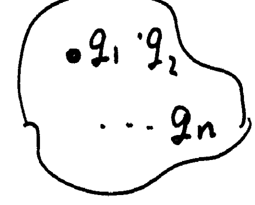
$$\Rightarrow \Phi_1 = \Phi_1 + \Phi_2 \Rightarrow \boxed{\Phi_2 = 0.}$$

if the closed surface enclose many charges $q_1, \cdots q_n . \Rightarrow$

$$\boxed{\oint \vec{E}\cdot d\vec{a} = 4\pi \sum_i q_i = 4\pi \int \rho\, dv}$$

integral form of Gauss's law

for continous distribution



Gauss's law $\longrightarrow$

$$\oint \vec{E}\cdot d\vec{a} = \int dv\, \nabla\cdot\vec{E} = 4\pi \int \rho\, dv \Rightarrow \boxed{\nabla\cdot\vec{E} = 4\pi\rho}$$

differential form.

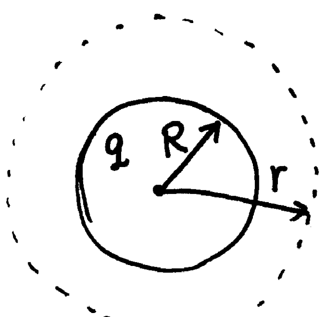
Gauss's law not only applies for electro-static fields, but also applies for dynamic electro-field. for the electro-static case, Gauss's law is equivalent to Coulumb's law. But Coulomb's law only applies for electro-static case, which is not as general as Gauss's law.

Proof : Coulomb's law $\longrightarrow$ Gauss's law

$$\vec{E}(\vec{r}) = \int d^3\vec{r}\,' \frac{\rho(\vec{r}\,')(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3} \Rightarrow \nabla\cdot\vec{E}(\vec{r}) = \int d^3\vec{r}\,' \rho(\vec{r}\,') \nabla\cdot\left(\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}\right)$$

$$= \int d^3\vec{r}\,' \rho(\vec{r}\,') \cdot 4\pi\delta(\vec{r}-\vec{r}\,') = 4\pi\rho\delta(\vec{r})$$

Application: electric field of a uniformally charged sphere



① for $r > R$, we draw a sphere with radius r.

exer: proof by symmetry argument $\longrightarrow$ $\vec{E}(r)$ is radial and has the same magnitude. Which symmetry ?

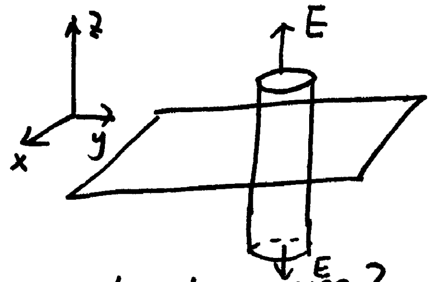
Then $\oint \vec{E}\cdot d\vec{r} = 4\pi q \Rightarrow \vec{E} = \frac{q}{r}\hat{e}_r$ which is the same as a point charge

② for $r < R$. again we draw a sphere with radius r, and again $\vec{E}$ is radial. The charge enclosed is $q\left(\frac{r}{R}\right)^3 \Rightarrow$

$$\oint \vec{E}\cdot d\vec{r} = 4\pi r^2 E = 4\pi q\left(\frac{r}{R}\right)^3 \Rightarrow E = \frac{qr}{R^2}\hat{e}_r$$

which is linear with r.

ex: an plane with charge density σ.



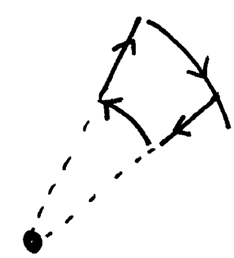
① please prove that electric field is parallel to z-direction everywell and independent of (x, y). which syms should we use?

② prove that at z and $-z$, $E_z(z) = -E_z(z)$. which sym do you want to use?

③ $\oint E \cdot dS = 4\pi q \Rightarrow 2ES = 4\pi S\sigma \Rightarrow \boxed{E = 2\pi\sigma}$ which is independent of the z.

§ The curl of $\vec{E}$. (for only electro-static field)

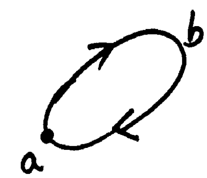
$$\vec{E} = \frac{q}{r^2}\hat{r} \text{. for a single charge}$$



$\int_a^b \vec{E} \cdot d\vec{l}$ and $d\vec{l} = \hat{r}\,dr + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}$

only the radial displacement contributes to work

$\int_a^b \vec{E} \cdot d\vec{l} = \int_{r_a}^{r_b} E(r)\,dr$, then for a closed loop



$$\oint \vec{E} \cdot d\vec{l} = \int_{r_a}^{r_b} E(r)dr + \int_{r_b}^{r_a} E(r)dr$$

$$\boxed{\oint \vec{E} \cdot d\vec{l} = 0}$$

$$\Rightarrow \boxed{\nabla \times \vec{E} = 0}$$

Please note! This is only correct for electro-static potential.

if for a general $\vec{E}$, which can be consider as $\vec{E} = \vec{E}_1 + \cdots \vec{E}_n$ (from a charge distri) we get the same results by linear-super position principle.