Lect 10 More on Maxwell equations — monopoles

- The Maxwell's equation in the free space is very symmetric.

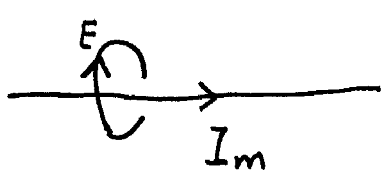
$$\nabla\cdot\vec{E}=0 \qquad \nabla\times\vec{E}=-\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$$
$$\nabla\cdot\vec{B}=0 \qquad \nabla\times\vec{B}=\frac{1}{c}\frac{\partial\vec{E}}{\partial t},$$

which are symetric under $\vec{E}\to\vec{B}$, $\vec{B}\to-\vec{E}$.

→ introducing magnetic monopole

$$\nabla\cdot\vec{E}=4\pi\rho_e \qquad \nabla\times\vec{E}=-4\pi\vec{j}_m-\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$$
$$\nabla\cdot\vec{B}=4\pi\rho_m \qquad \nabla\times\vec{B}=4\pi\vec{j}_e+\frac{1}{c}\frac{\partial\vec{E}}{\partial t}$$

- imagine a world with only magnetic monopole but not electric charge the monopole currents generate circular electric-static current but follows the left-hand law. Why? There should be another version of magneto-electric induction which satisfies Lentz law.



imagine E-flux is increasing with time, what's the induced magnetic-(mmf) from $\nabla\times\vec{B}=\frac{1}{c}\frac{\partial\vec{E}}{\partial t}$? (as plotted in the left)

The monopole current $\vec{B}$, generate produces induced by $\vec{E}$ field against the original increasing of $\vec{E}$ field.

- So it is counter-mmf.

The signs of RHS of the last two Maxwell equations have deep meaning. g

fixed by the "displacement current"

$$\nabla \times \vec{B} = +4\pi \vec{j}_e + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$$

let's fix it by experiments of Ampere's law

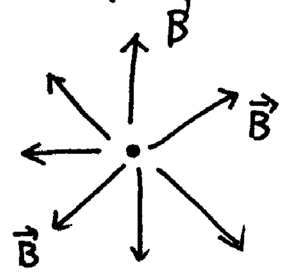
magnetic-version of Lentz law "counter-mmf"

$$\nabla \times \vec{E} = -4\pi \vec{j}_m - \frac{1}{c}\frac{\partial \vec{B}}{\partial t}$$

Lentz law "counter emf".

§ magnetic monopole (a point electric charge's motion, cf Prob 5.43)

$$\vec{B} = \frac{g\hat{r}}{r^2}$$



Summarize:

$$m\frac{d^2\vec{r}}{dt^2} = \frac{e}{c}\vec{v}\times\vec{B} = \frac{eg}{cr^3}\left(\frac{d\vec{r}}{dt}\times\vec{r}\right)$$

1° Kinetic energy is conserved: $\frac{m}{2}\frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right)^2 = 0$.

2° no-closed orbit: $\vec{r}\cdot\frac{d^2\vec{r}}{dt^2} = 0 \Rightarrow \frac{1}{2}\frac{d^2}{dt^2}(r^2) = \frac{d}{dt}\left(\vec{r}\cdot\frac{d\vec{r}}{dt}\right)$

$$= \left(\frac{d\vec{r}}{dt}\right)^2 + \vec{r}\cdot\frac{d^2\vec{r}}{dt^2} = v^2$$

$$\Rightarrow \begin{cases} r^2 = v^2t^2 + b^2 \\ \vec{r}\cdot\vec{v} = \frac{1}{2}\frac{d}{dt}r^2 = v^2 t \end{cases}$$

the linear term on "t" can be absorbed by shift the origin of t.

r^2, and $\vec{v}\cdot\vec{r}$ increases with t, electric charge falls down from infinity into the monopole, approach a minimal distance b, and flee away.

3° The usual definition $\vec{\mathcal{L}} = \vec{r} \times m\vec{v}$ is not conserved because Lorentz force is not a center-force field.

$$\frac{d}{dt}\vec{\tilde{\mathcal{L}}} = \vec{r} \times m\frac{d\vec{v}}{dt} = \frac{eg}{r^3}\,\vec{r} \times (\vec{v} \times \vec{r}) = \frac{eg}{mcr^3}\,\vec{\tilde{\mathcal{L}}} \times \vec{r}$$

$$\frac{d\tilde{\mathcal{L}}^2}{dt} = \vec{\tilde{\mathcal{L}}} \cdot \frac{d\vec{\tilde{\mathcal{L}}}}{dt} = \frac{eg}{mcr^3}\,\vec{\tilde{\mathcal{L}}} \cdot (\vec{\tilde{\mathcal{L}}} \times \vec{r}) = 0 \Rightarrow |\tilde{\mathcal{L}}| = mvb$$

Nevertheless we can still define $\vec{\mathcal{L}} = \vec{r} \times m\vec{v} - \frac{eg}{c}\frac{\vec{r}}{r}$ which remains conserved

↑ like spin.

$$\frac{d}{dt}\vec{\tilde{\mathcal{L}}} = \frac{eg}{r^3}\,\vec{r} \times \left(\frac{d\vec{r}}{dt} \times \vec{r}\right) = \frac{eg}{r^3}\left[r^2\frac{d\vec{r}}{dt} - \left(\vec{r} \cdot \frac{d\vec{r}}{dt}\right)\vec{r}\right]$$

$$= \frac{eg}{r^3}\left[r^2\frac{d\vec{r}}{dt} - r\frac{dr}{dt}\right]\vec{r} = \frac{eg}{r^3} \cdot r^3\left[\frac{1}{r}\frac{d\vec{r}}{dt} - \left(\frac{1}{r^2}\frac{dr}{dt}\right)\vec{r}\right] = eg\frac{d}{dt}\left(\frac{\vec{r}}{r}\right)$$

$$\Rightarrow \frac{d}{dt}\left(\vec{\tilde{\mathcal{L}}} - eg\frac{\vec{r}}{r}\right) = \frac{d}{dt}\vec{\mathcal{L}} = 0 .$$

$$\vec{\tilde{\mathcal{L}}} \perp \frac{eg}{c}\frac{\vec{r}}{r} \qquad \Rightarrow \qquad \boxed{L^2 = (mvb)^2 + \left(\frac{eg}{c}\right)^2}$$

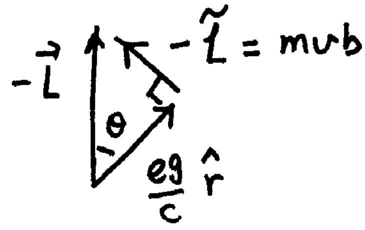
4° the integral of motion

$$E = \frac{m}{2}v^2 = \frac{m}{2}\dot{r}^2 + \frac{\tilde{\mathcal{L}}^2}{2mr^2} = \frac{1}{2}m\dot{\vec{r}}^2 + \frac{\vec{\mathcal{L}}^2 - \left(\frac{eg}{c}\right)^2}{2mr^2} .$$

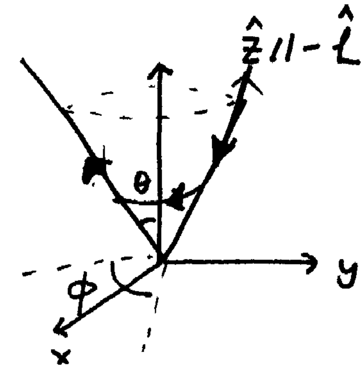
5° sketch of motion.

Set the direction of $-\vec{\mathcal{L}}$ as the z-axis. Acording to $\vec{\mathcal{L}} \cdot \hat{r} = -\frac{eg}{c}$

$\Rightarrow$ particles lies in a cone as



$$\Rightarrow \operatorname{ctg}\theta = \frac{eg/c}{\tilde{L}} = \frac{eg}{mvbc}$$



$\vec{\tilde{L}}$ processes around $\vec{L}$.

$$\boxed{L = \frac{eg}{c\cos\theta}}, \quad \cancel{mvb = \frac{eg}{c}}$$

6° the motion of the azimuthal angle ϕ.

$$\vec{L}\cdot\hat{e}_\theta = -L\sin\theta = \left[m(\vec{r}\times\vec{v}) - \frac{eg}{c}\hat{r}\right]\cdot\hat{e}_\theta = m(\vec{r}\times\vec{v})\cdot\hat{e}_\theta$$

$$= m(r\hat{r} \times r\sin\theta\,\dot{\phi}\,\hat{e}_\phi)\cdot\hat{e}_\theta = -m r^2 \sin\theta\,\dot{\phi}$$

$$\Rightarrow \quad \dot{\phi} = +\frac{L}{m}\frac{1}{r^2} = \frac{eg}{c\cos\theta}\frac{1}{r^2}$$

7° the trajectory: $\quad v^2 = \dot{r}^2 + r\dot{\theta}^2 + (r\sin\theta\,\dot{\phi})^2 = \dot{r}^2 + \left(r\sin\theta\frac{L}{m}\frac{1}{r^2}\right)^2$

$$\Rightarrow \quad \dot{r}^2 = v^2 - \left(\frac{L}{m}\right)^2 \frac{\sin^2\theta}{r^2}$$

$$\Rightarrow \quad \frac{dr}{d\phi} = \frac{\dot{r}}{\dot{\phi}} = \frac{\sqrt{v^2 - \left(\frac{L}{m}\right)^2\frac{\sin^2\theta}{r^2}}}{\left(\frac{L}{m}\right)\frac{1}{r^2}} = r\sqrt{\left(\frac{vr}{L/m}\right)^2 - \sin^2\theta}$$

$$\text{or} \quad \boxed{\frac{d\phi}{dr} = \frac{1}{r\sqrt{\left(\frac{vr}{L/m}\right)^2 - \sin^2\theta}}}$$

at $r\to\infty$, the motion is almost a straight line along $\hat{e}_r$ with fixed (θ, ϕ).

It is deflected and sent to infinity again!

§ Vector potential for magnetic monopoles

rigorously speaking, because $\nabla\cdot\vec B(r) = 4\pi\,\delta^3(\vec r) \neq 0$, there should be no regular A-potential can produce $\nabla\times\vec A = \frac{g\,\vec r}{r^3}$. Nevertheless, we still need A for quantum-mechanics $H = \frac{(\vec P - \frac{e}{c}\vec A)^2}{2m}$ to describle a charged particle moving in a monopole field.

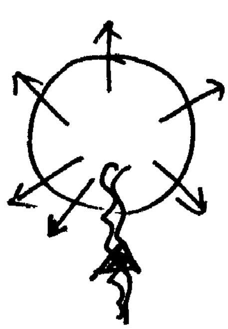
A convenient choice is that $\vec A(\vec r) = \frac{g}{r}\,\frac{\sin\theta}{1+\cos\theta}\,\hat e_\varphi$.

check
$$\nabla\times\vec A = \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(\sin\theta\,A_\phi) - \frac{\partial A_\theta}{\partial\varphi}\right)\hat e_r + \left(\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\varphi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\varphi)\right)\hat e_\theta$$
$$+\left(\frac{1}{r}\frac{\partial}{\partial r}(rA_\theta) - \frac{1}{r}\frac{\partial A_r}{\partial\theta}\right)\hat e_\varphi$$

$$\Rightarrow \nabla\times\vec A = \frac{g}{r^2}\,\frac{1}{\sin\theta}\,\frac{\partial}{\partial\theta}\left[2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\,\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\right]\hat e_r = \frac{g}{r^2}\,\frac{1}{\sin\theta}\,2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\,\hat e_r$$
$$= \frac{g}{r^2}\,\hat e_r .$$

How can a $\nabla\times\vec A$ describes a ~~mon~~ field has divergence?

$\vec A(r) = \frac{g}{r}\,\frac{\sin\theta}{1+\cos\theta}\,\hat e_\varphi$ has a singlular point at south pole $\theta = -\pi$.



Let's choose a infibitismal loop around south pole. If A were regular we will get zero flux as the loop shrinks to zero. but now

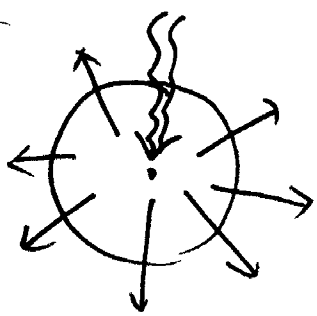
$$\oint\vec A\cdot d\vec l = -2\pi g\,\frac{R}{R}\,\sin\theta\cdot tg\frac{\theta}{2}\,\Big|_{\theta\to\pi} = -2\pi g\,2\sin^2\frac{\theta}{2} = -4\pi g$$

$\Rightarrow$ rigorously speaking, $\nabla \times \vec{A} = \frac{g\vec{r}}{r^3} - 4\pi\delta(\vec{r}\text{ at southpole})\,\hat{e}_r$.

So the total flux remains zero. The singular part is called dirac string!! Should electrons see this string? By the consistency of requirement of theory, we should not let Dirac string visible.

I can chose another gauge such that the Dirac string is at the north pole. $\tilde{A} = \frac{g}{R}\,\mathrm{ctg}\frac{\theta}{2}\,\hat{e}_\varphi$, we can check that

$$\nabla \times \tilde{A} = \frac{g}{r^3}\vec{r} - 4\pi\delta(\vec{r}\text{ at north pole})\,\hat{e}_r .$$



The location of Dirac string cannot be specified uniquely. The only way for consistency it that for an electron move around it, it picks up a phase of $2n\pi$, thus its unvisible.

$$\frac{eg}{\hbar c}\oint A\cdot dn = \frac{eg}{\hbar c}\cdot 4\pi = 2n\pi$$

$$\Rightarrow eg = \frac{n}{2}\cdot \hbar c$$, the product of eg should be quantized!