PHYS 100B (Prof. Congjun Wu)
Solution to HW 5

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Problem 1 (Griffiths 6.26)

At the interface between two linear magnetic materials, the magnetic field lines band. Show that $\tan\theta_2/\tan\theta_1 = \mu_2/\mu_1$, assuming there is no free current at the boundary. Compare Eq. 4.68.

Solution: At the interface, (1)$B_1^{\perp} = B_2^{\perp}$, (2) $\mathbf{B}_1^{\parallel} = \mathbf{B}_2^{\parallel} + \mu_0(\mathbf{K}\times\hat{\mathbf{n}}) \Rightarrow \mathbf{H}_1^{\parallel} = \mathbf{H}_2^{\parallel} + \mathbf{K}_f \times \hat{\mathbf{n}}$ (Since $\frac{1}{\mu_0}\mathbf{B} = \mathbf{H}+\mathbf{M}$). No free current at the boundary $\Rightarrow \mathbf{H}_1^{\parallel} = \mathbf{H}_2^{\parallel} \Rightarrow \frac{1}{\mu_1}\mathbf{B}_1^{\parallel} = \frac{1}{\mu_2}\mathbf{B}_2^{\parallel}$.

$$\tan\theta_2/\tan\theta_1 = \frac{|\mathbf{B}_2^{\parallel}|}{B_2^{\perp}}/\frac{|\mathbf{B}_1^{\parallel}|}{B_1^{\perp}} = |\mathbf{B}_2^{\parallel}|/|\mathbf{B}_1^{\parallel}| = \mu_2/\mu_1.$$

Problem 2 (Griffiths 6.27)

A magnetic dipole $\mathbf{m}$ is imbedded at the center of a sphere (radius R) of linear magnetic material with permeability μ. Show that the magnetic field inside the sphere ($0 < r \le R$) is

$$\frac{\mu}{4\pi}\left\{\frac{1}{r^3}\left[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}\right] + \frac{2(\mu_0-\mu)\mathbf{m}}{(2\mu_0+\mu)R^3}\right\}.$$

What is the field outside the sphere?

Solution: In view of Eq. 6.33, the volume bound current density is proportional to the free current density

$$\mathbf{J}_b = \nabla\times\mathbf{M} = \nabla\times(\chi_m\mathbf{H}) = \chi_m\mathbf{J}_f.$$

There is a bound dipole at the center

$$\mathbf{m}_b = \chi_m\mathbf{m}.$$

$\Rightarrow$ Net dipole moment at the center is

$$\mathbf{m}_{center} = \mathbf{m}+\mathbf{m}_b = (1+\chi_m)\mathbf{m} = \frac{\mu}{\mu_0}\mathbf{m}.$$

$\Rightarrow$ (Eq. 5.87) Magnetic field produced by the dipole:

$$\mathbf{B}_{\substack{center\\dipole}} = \frac{\mu}{4\pi}\left\{\frac{1}{r^3}\left[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}\right]\right\}.$$

Magnetic field produced by the bound surface current $\mathbf{K}_b$ at $r = R$: (No volume bound current $\mathbf{J}_b$ since no free current $\mathbf{J}_f$ flowing through the material)

$$\mathbf{B}_{\substack{surface\\current}} = A\mathbf{m}.$$

The magnetization

$$\mathbf{M} = \chi_m\mathbf{H} = \frac{\chi_m}{\mu}\mathbf{B} = \frac{\chi_m}{4\pi}\left\{\frac{1}{r^3}\left[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}\right]\right\} + \frac{\chi_m}{\mu}A\mathbf{m}.$$

$\Rightarrow$ bound current density

$$\mathbf{J}_b = \nabla\times\mathbf{M} = \mathbf{0}.$$

$$\mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{r}} = \left(\frac{\chi_m}{4\pi} \left\{ \frac{1}{r^3} \left[-\mathbf{m} \times \hat{\mathbf{r}} \right] \right\} + \frac{\chi_m}{\mu} A\mathbf{m} \times \hat{\mathbf{r}} \right) |_{r=R} = \chi_m m \left(-\frac{1}{4\pi R^3} + \frac{A}{\mu} \right) \sin\theta \hat{\phi}.$$

This is the surface current produced by a spinning sphere: $\mathbf{K}_b = \sigma\omega R \sin\theta\hat{\phi}$, $\sigma\omega R = \chi_m m \left(\frac{A}{\mu} - \frac{1}{4\pi R^3} \right)$. $\Rightarrow$

$$B_{\substack{surface\\current}} = \frac{2}{3}\mu_0 \sigma\omega R = \frac{2}{3}\mu_0 \chi_m m \left(\frac{A}{\mu} - \frac{1}{4\pi R^3} \right) = A.$$

and $\chi_m = \frac{\mu}{\mu_0} - 1$, $\Rightarrow$

$$A = \frac{\mu}{4\pi} \frac{2(\mu_0 - \mu)}{R^3(2\mu_0 + \mu)}.$$

$\Rightarrow$

$$\mathbf{B} = \frac{\mu}{4\pi} \left\{ \frac{1}{r^3} \left[3(\mathbf{m} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m} \right] + \frac{2(\mu_0 - \mu)\mathbf{m}}{R^3(2\mu_0 + \mu)} \right\}.$$

The field outside the sphere is that of the central dipole plus the dipole field from the surface current,

$$\mathbf{m}_{\substack{surface\\current}} = \frac{4\pi R^3}{3}(\sigma\omega R) = \frac{4\pi R^3}{3} \frac{3}{2\mu_0} \mathbf{B}_{\substack{surface\\current}} = \frac{4\pi R^3}{3} \frac{3}{2\mu_0} \frac{\mu}{4\pi} \frac{2(\mu_0 - \mu)\mathbf{m}}{R^3(2\mu_0 + \mu)} = \frac{\mu(\mu_0 - \mu)\mathbf{m}}{\mu_0(2\mu_0 + \mu)}.$$

$\Rightarrow$

$$\mathbf{m}_{tot} = \frac{\mu\mathbf{m}}{\mu_0} + \frac{\mu(\mu_0 - \mu)\mathbf{m}}{\mu_0(2\mu_0 + \mu)} = \frac{3\mu\mathbf{m}}{(2\mu_0 + \mu)}.$$

and hence the field for $r > R$ is

$$\begin{aligned}
\mathbf{B} &= \frac{\mu}{4\pi} \left\{ \frac{1}{r^3} \left[3(\mathbf{m}_{tot} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}_{tot} \right] + \frac{2(\mu_0 - \mu)\mathbf{m}_{tot}}{R^3(2\mu_0 + \mu)} \right\} \\
&= \frac{\mu}{4\pi} \frac{3\mu}{(2\mu_0 + \mu)} \left\{ \frac{1}{r^3} \left[3(\mathbf{m} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m} \right] + \frac{2(\mu_0 - \mu)\mathbf{m}}{R^3(2\mu_0 + \mu)} \right\}.
\end{aligned}$$

Problem 3 (Griffiths 7.60)
(a) Show that Maxwell's equations with magnetic charge (Eq. 7.43) are invariant under the duality transformation

$$\begin{aligned}
\mathbf{E}' &= \mathbf{E}\cos\alpha + c\mathbf{B}\sin\alpha, \\
c\mathbf{B}' &= c\mathbf{B}\cos\alpha - \mathbf{E}\sin\alpha, \\
cq'_e &= cq_e \cos\alpha + q_m \sin\alpha, \\
q'_m &= q_m \cos\alpha - cq_e \sin\alpha.
\end{aligned}$$

where $c = 1/\sqrt{\epsilon_0\mu_0}$ and α is an arbitrary rotation angle in $\mathbf{E}/\mathbf{B}$-space. Charge and current densities transform in the same way as q_e and q_m. [This means, in particular, that if you know the fields produced by a configuration of electric charge, you can immediately (using $\alpha = 90°$) write down the fields produced by the corresponding arrangement of magnetic charge.]

Solution:

$$\begin{aligned}
\nabla\cdot\mathbf{E}' &= \nabla\cdot\mathbf{E}\cos\alpha + c\nabla\cdot\mathbf{B}\sin\alpha = \frac{1}{\epsilon_0}\rho_e\cos\alpha + c\mu_0\rho_m\sin\alpha\\
&= \frac{1}{\epsilon_0}\left(\rho_e\cos\alpha + \frac{1}{c}\rho_m\sin\alpha\right) = \frac{1}{\epsilon_0}\rho_e'.\\
\nabla\cdot\mathbf{B}' &= \nabla\cdot\mathbf{B}\cos\alpha - \frac{1}{c}\nabla\cdot\mathbf{E}\sin\alpha = \mu_0\rho_m\cos\alpha - \frac{1}{c}\frac{1}{\epsilon_0}\rho_e\sin\alpha\\
&= \mu_0\left(\rho_m\cos\alpha - c\rho_e\sin\alpha\right) = \mu_0\rho_m'.\\
\nabla\times\mathbf{E}' &= \nabla\times\mathbf{E}\cos\alpha + c\nabla\times\mathbf{B}\sin\alpha = \left(-\mu_0\mathbf{J}_m - \frac{\partial\mathbf{B}}{\partial t}\right)\cos\alpha + c\left(\mu_0\mathbf{J}_e + \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}\right)\sin\alpha\\
&= -\mu_0\left(\mathbf{J}_m\cos\alpha - c\mathbf{J}_e\sin\alpha\right) - \frac{\partial}{\partial t}\left(\mathbf{B}\cos\alpha - \frac{1}{c}\mathbf{E}\sin\alpha\right)\\
&= -\mu_0\mathbf{J}_m' - \frac{\partial\mathbf{B}'}{\partial t},\\
\nabla\times\mathbf{B}' &= \nabla\times\mathbf{B}\cos\alpha - \frac{1}{c}\nabla\times\mathbf{E}\sin\alpha = \left(\mu_0\mathbf{J}_e + \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}\right)\cos\alpha - \frac{1}{c}\left(-\mu_0\mathbf{J}_m - \frac{\partial\mathbf{B}}{\partial t}\right)\sin\alpha\\
&= \mu_0\left(\mathbf{J}_e\cos\alpha + \frac{1}{c}\mathbf{J}_m\sin\alpha\right) + \mu_0\epsilon_0\frac{\partial}{\partial t}\left(\mathbf{E}\cos\alpha + c\mathbf{B}\sin\alpha\right)\\
&= \mu_0\mathbf{J}_e' + \mu_0\epsilon_0\frac{\partial\mathbf{E}'}{\partial t}.
\end{aligned}$$

(b) Show that the force Law (Prob. 7.35)

$$\mathbf{F} = q_e(\mathbf{E} + \mathbf{v}\times\mathbf{B}) + q_m(\mathbf{B} - \frac{1}{c^2}\mathbf{v}\times\mathbf{E}),$$

is also invariant under the duality transformation.]

Solution:

$$\begin{aligned}
\mathbf{F}' &= q_e'(\mathbf{E}' + \mathbf{v}\times\mathbf{B}') + q_m'(\mathbf{B}' - \frac{1}{c^2}\mathbf{v}\times\mathbf{E}')\\
&= \left(q_e\cos\alpha + \frac{1}{c}q_m\sin\alpha\right)\left[(\mathbf{E}\cos\alpha + c\mathbf{B}\sin\alpha) + \mathbf{v}\times\left(\mathbf{B}\cos\alpha - \frac{1}{c}\mathbf{E}\sin\alpha\right)\right]\\
&\quad + (q_m\cos\alpha - cq_e\sin\alpha)\left[\left(\mathbf{B}\cos\alpha - \frac{1}{c}\mathbf{E}\sin\alpha\right) - \frac{1}{c^2}\mathbf{v}\times(\mathbf{E}\cos\alpha + c\mathbf{B}\sin\alpha)\right]\\
&= q_e\left[(\mathbf{E}\cos\alpha\cos\alpha + c\mathbf{B}\sin\alpha\cos\alpha) + \mathbf{v}\times\left(\mathbf{B}\cos\alpha\cos\alpha - \frac{1}{c}\mathbf{E}\sin\alpha\cos\alpha\right)\right]\\
&\quad + q_e\left[-(\mathbf{B}c\sin\alpha\cos\alpha - \mathbf{E}\sin\alpha\sin\alpha) + \mathbf{v}\times\left(\frac{1}{c}\mathbf{E}\sin\alpha\cos\alpha + \mathbf{B}\sin\alpha\sin\alpha\right)\right]\\
&\quad + q_m\left[\left(\mathbf{E}\frac{1}{c}\sin\alpha\cos\alpha + \mathbf{B}\sin\alpha\sin\alpha\right) + \mathbf{v}\times\left(\frac{1}{c}\mathbf{B}\sin\alpha\cos\alpha - \frac{1}{c^2}\mathbf{E}\sin\alpha\sin\alpha\right)\right]\\
&\quad + q_m\left[\left(\mathbf{B}\cos\alpha\cos\alpha - \frac{1}{c}\mathbf{E}\sin\alpha\cos\alpha\right) - \mathbf{v}\times\left(\frac{1}{c^2}\mathbf{E}\cos\alpha\cos\alpha + \frac{1}{c}\mathbf{B}\sin\alpha\cos\alpha\right)\right]\\
&= q_e\left(\mathbf{E} + \mathbf{v}\times\mathbf{B}\right) + q_m\left[\mathbf{B} - \mathbf{v}\times\frac{1}{c^2}\mathbf{E}\right] = \mathbf{F}.
\end{aligned}$$