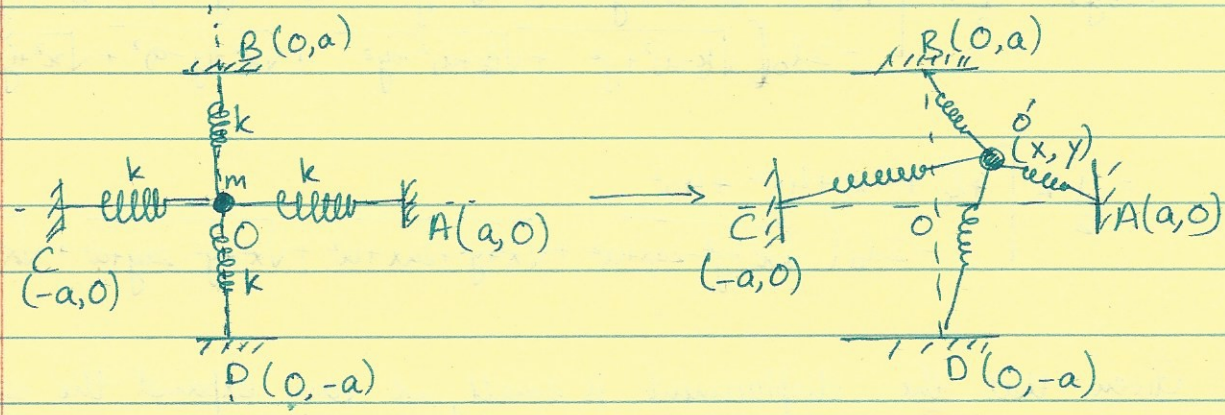
PHYS 110A HW #6

5.19



Let AC & BD define the x & y axis respectively. In equilibrium, the mass is at O and $OA = OB = OC = OD = a$.

Suppose the mass is displaced to the point O' whose coordinates are (x, y), where $x, y \ll a$ & $x, y \ll l_0$.

The potential energy of the system is given by

$$U(x,y) = \frac{1}{2} k \left[(O'A - l_0)^2 + (O'B - l_0)^2 + (O'C - l_0)^2 + (O'D - l_0)^2 \right]$$

$$= \frac{1}{2} k \left[4l_0^2 + O'A^2 + O'B^2 + O'C^2 + O'D^2 - 2l_0 (O'A + O'B + O'C + O'D) \right]$$

Now

$$O'A = \sqrt{(x-a)^2 + y^2}$$

$$O'B = \sqrt{x^2 + (y-a)^2}$$

$$O'C = \sqrt{(x+a)^2 + y^2}$$

$$O'D = \sqrt{x^2 + (y+a)^2}$$

Plugging these into the expression for potential energy,

$$U(x,y) = \frac{1}{2}k\left[4l_0^2 + (x-a)^2+y^2+(x+a)^2+y^2+x^2+(y-a)^2+x^2+(y+a)^2 - 2l_0\left(\sqrt{(x-a)^2+y^2}+\sqrt{(x+a)^2+y^2}+\sqrt{x^2+(y-a)^2}+\sqrt{x^2+(y+a)^2}\right)\right]$$

$$= \frac{1}{2}k\left[4l_0^2+4x^2+4y^2+4a^2 - 2l_0\left(\sqrt{x^2+y^2-2ax+a^2}+\sqrt{x^2+y^2+2ax+a^2}+\sqrt{x^2+y^2-2ay+a^2}+\sqrt{x^2+y^2+2ay+a^2}\right)\right]$$

Given that the displacement is small, we can expand the square root terms using taylor series. Since, we are looking at potential due to springs, we need only expand to second order in x & y.

~~For example~~

Use $(1+\delta)^n \approx 1+n\delta+\frac{n(n-1)}{2}\delta^2+\frac{n(n-1)(n-2)}{6}\delta^3 \ldots\ldots$

$$\approx 1+n\delta+\frac{n(n-1)}{2}\delta^2 \quad \text{to second order for } \delta \ll 1.$$

Using this formula here

$$\sqrt{a^2+x^2+y^2-2ax} = a\left(1+\frac{x^2+y^2}{a^2}-\frac{2x}{a}\right)^{1/2}$$

$$\approx a\left[1+\frac{1}{2}\left(\frac{x^2+y^2}{a^2}-\frac{2x}{a}\right)-\frac{1}{8}\left(\frac{x^2+y^2}{a^2}-\frac{2x}{a}\right)^2\right]$$

$$= a\left[1+\frac{x^2+y^2}{2a^2}-\frac{x}{a}-\frac{1}{8}\left(\frac{x^4}{a^4}+\frac{y^4}{a^4}+\frac{4x^2}{a^2}+\frac{2x^2y^2}{a^4}-\frac{4x^3}{a^3}-\frac{4xy^2}{a^3}\right)\right]$$

Note that we only want to retain ~~the~~ terms upto order x^2, y^2.

So

$$\sqrt{a^2+x^2+y^2-2ax} = a\left[1+\frac{x^2}{2a^2}+\frac{y^2}{2a^2}-\frac{x}{a}-\frac{x^2}{2a^2}\right]$$

$$= a+\frac{y^2}{2a}-x$$

Similarly,

$$\sqrt{(x+a)^2+y^2} \approx a + \frac{y^2}{2a} + x$$

$$\sqrt{x^2+(y-a)^2} \approx a + \frac{x^2}{2a} - y$$

$$\sqrt{x^2+(y+a)^2} \approx a + \frac{x^2}{2a} + y$$

$$U(x,y) = \frac{1}{2}k\left[4l_0^2 + 4a^2 + 4x^2 + 4y^2 - 2l_0\left(4a + \frac{x^2}{a} + y^2/a\right)\right]$$

$$= \frac{1}{2}k\left[4a^2 + 4l_0^2 - 8l_0 a + 4x^2 + 4y^2 - \frac{2l_0}{a}x^2 - \frac{2l_0}{a}y^2\right]$$

$$= \frac{1}{2}k\left[4(l_0-a)^2 + 4(1 - l_0/2a)(x^2+y^2)\right]$$

$$U(x,y) = 2k(l_0-a)^2 + 2k\left(1-\frac{l_0}{2a}\right)(x^2+y^2)$$

But $x^2+y^2 = r^2$

$$U(x,y) = \underbrace{2k(l_0-a)^2}_{\text{constant}} + 2k\left(1-\frac{l_0}{2a}\right)r^2$$

So, $U = U_0 + \frac{1}{2}k'r^2$ where $\boxed{k' = 4k\left(1-\frac{l_0}{2a}\right)}$

The force is given by the derivative of U

$$\bar{F} = -\nabla U = -\frac{\partial U}{\partial r}\hat{r} = -k' r\hat{r} = -k'\vec{r}$$

$$\boxed{\vec{F} = -4k\left(1-\frac{l_0}{2a}\right)\vec{r}}$$

5.32 (a) $x(t) = e^{-\beta t}\left[B_1 \cos\omega_1 t + B_2 \sin\omega_1 t\right]$

Given $x(t=0) = x_0$ & $\dot{x}(t=0) = 0$,

$\Rightarrow \boxed{x_0 = B_1}$

$$\dot{x}(t) = e^{-\beta t}\omega_1\left[-B_1 \sin\omega_1 t + B_2 \cos\omega_1 t\right] - \beta e^{-\beta t}\left[B_1 \cos\omega_1 t + B_2 \sin\omega_1 t\right]$$

$$0 = B_2\omega_1 - \beta B_1$$

$$\Rightarrow \boxed{B_2 = \frac{\beta}{\omega_1} x_0}$$

So, $\boxed{x(t) = x_0 e^{-\beta t}\left[\cos\omega_1 t + \frac{\beta}{\omega_1}\sin\omega_1 t\right]}$,

(b) If $\beta \to \omega_0$, $\omega_1 \to 0$

$\Rightarrow \cos\omega_1 t \to 1$

$\frac{\sin\omega_1 t}{\omega_1}$ is indeterminate ~~but~~ but we know that

$\lim_{x\to 0}\frac{\sin x}{x} = 1.$

$$\Rightarrow \frac{\sin\omega_1 t}{\omega_1} = t\left(\frac{\sin\omega_1 t}{\omega_1 t}\right) \to t$$

$$\Rightarrow x(t) = x_0 e^{-\omega_0 t}\left[1 + \beta t\right] = x_0 e^{-\omega_0 t}\left(1 + \omega_0 t\right)$$

$$\boxed{x(t) = x_0 e^{-\omega_0 t}\left(1 + \omega_0 t\right)}$$

5.38 $x(t) = A\cos(\omega t - \delta) + e^{-\beta t}\left[B_1 \cos\omega_1 t + B_2 \sin\omega_1 t\right]$

Now, $A^2 = \dfrac{f_0^2}{(\omega_0^2-\omega^2)^2 + 4\beta^2\omega^2}$ $\quad \delta = \tan^{-1}\left(\dfrac{2\beta\omega}{\omega_0^2-\omega^2}\right)$

$\omega_1 = \sqrt{\omega_0^2 - \beta^2}$

Given $\omega = \omega_0 = 1$, $\beta = 0.1$ & $f_0 = 0.4$

$$A^2 = \frac{(.4)^2}{4(.1)^2(1)^2} \Rightarrow A = \frac{.4}{2\times .1} = 2$$

$$\boxed{\begin{array}{l} A = 2 \\ \delta = \tan^{-1}\left(\dfrac{2\times .1\times 1}{0}\right) = \dfrac{\pi}{2} \end{array}}$$

$$\boxed{\omega_1 = \sqrt{1-(.1)^2} = \sqrt{0.99}}$$

$$\Rightarrow x(t) = 2\cos(t-\pi/2) + e^{-.1t}\left[B_1\cos(\sqrt{.99}\,t) + B_2\sin(\sqrt{.99}\,t)\right]$$

$$= 2\sin t + e^{-.1t}\left[B_1\cos(\sqrt{.99}\,t) + B_2\sin(\sqrt{.99}\,t)\right]$$

Given $x_0 = 0 = x(t=0)$

$v_0 = 6 = \dot{x}(t=0)$,

$$\boxed{0 = B_1}$$

$$\dot{x}(t) = 2\cos t + e^{-.1t}(\sqrt{.99})\left[B_2\cos(\sqrt{.99}\,t)\right] - (.1)e^{-.1t}\left[B_2\sin(\sqrt{.99}\,t)\right]$$

$$\dot{x}(t=0) = v_0 = 6 = 2 + (\sqrt{.99})B_2 \Rightarrow \boxed{B_2 = \frac{4}{\sqrt{.99}}}$$

$$\Rightarrow \quad \boxed{x(t) = 2\sin t + \frac{4}{\sqrt{0.99}}\, e^{-0.1t} \sin\left(\sqrt{.99}\, t\right)}$$

5.45 $F(t) = F_0 \cos \omega t$

The equation for a damped driven oscillator is given by

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f_0 \cos\omega t$$

where $f_0 = \frac{F_0}{m}$

The solution to the equation is then given by

$$x(t) = A\cos(\omega t - \delta) + e^{-\beta t}\left[B_1 \cos\omega_1 t + B_2 \sin\omega_1 t\right]$$

Since the transient part of the solution decays exponentially, it does not contribute to the long time average. Therefor, we need only focus on the harmonic part of the solution, ie

$$x(t) = A\cos(\omega t - \delta) = A\cos\omega t \cos\delta + A\sin\omega t \sin\delta$$

$$\Rightarrow v(t) = \dot{x}(t) = -A\omega\left[\sin\omega t \cos\delta - \sin\delta \cos\omega t\right]$$

(a) $P(t) = F(t) . v(t)$

$$= -F_0 A\omega \cos\omega t\left[\sin\omega t \cos\delta - \sin\delta \cos\omega t\right]$$

The average power

$$\langle P \rangle = \lim_{T\to\infty} \frac{1}{T}\int_0^T P(t)\,dt$$

But since P is periodic with period $\frac{2\pi}{\omega}$,

$$\langle P \rangle = \frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} P(t)\,dt$$

$$= -\frac{F_0 A\omega^2}{2\pi}\int_0^{\frac{2\pi}{\omega}} \left[(\sin\omega t\cos\omega t)\cos\delta - (\cos^2\omega t \sin\delta)\right]dt$$

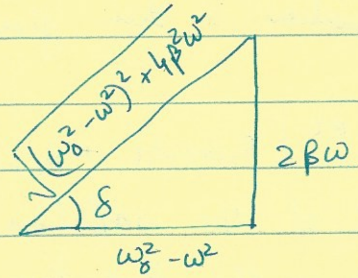
Note that $\int_0^{\frac{2\pi}{\omega}} \sin\omega t \cos\omega t = 0$ and $\int_0^{\frac{2\pi}{\omega}} \cos^2\omega t = \frac{1}{2}\left(\frac{2\pi}{\omega}\right) = \frac{\pi}{\omega}$

$\Rightarrow$

$$\langle P \rangle = \frac{F_0 A\omega^2}{2\cancel{\pi}} \sin\delta \left(\frac{\cancel{\pi}}{\omega}\right) = \frac{F_0 A\omega}{2}\sin\delta$$

Now

$$A = \frac{F_0/m}{\sqrt{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2}} \quad \& \quad \tan\delta = \frac{2\beta\omega}{\sqrt{(\omega_0^2-\omega^2)^2}}$$



Plugging in values of F_0 & $\sin\delta$,

$$\langle P\rangle = \frac{A\omega}{\cancel{2}}\left[mA\cancel{\sqrt{4\beta^2\omega^2+(\omega_0^2-\omega^2)^2}}\right]\left[\frac{\cancel{2}\beta\omega}{\cancel{\sqrt{4\beta^2\omega^2+(\omega_0^2-\omega^2)^2}}}\right]$$

$$\boxed{\langle P\rangle = mA^2\omega^2\beta}$$

This is the average power delivered by the force $F(t)$

(b) The resistive force is given by

$$f_r = -2m\beta\dot{x} = -2m\beta v$$

$\Rightarrow$ ~~Power~~ ~~dE/dt~~ Energy loss due to resistive force $\cancel{(P(t))}$ $P_1(t)$

$$P_1(t)\,\cancel{\dot{E}(t)} = f_r\, v(t)$$

$$P_1(t)\,\cancel{\dot{E}(t)} = -2m\beta v^2$$

$$= -2m\beta A^2\omega^2\left(\cos\omega t\sin\delta - \sin\omega t\cos\delta\right)^2$$

$$\langle P_1(t)\rangle\,\cancel{\langle\dot{E}(t)\rangle} = \frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} P_1(t)\,dt$$

$$= -\frac{\omega}{\cancel{2}\pi}\int_0^{\frac{2\pi}{\omega}} \cancel{2}m\beta A^2\omega^2\left(\cos\omega t\sin\delta - \sin\omega t\cos\delta\right)^2 dt$$

$$= -m\beta A^2\omega^2\int_0^{\frac{2\pi}{\omega}} \frac{\omega}{\pi}\sin^2(\omega t-\delta)\,dt$$

~~let $\omega t - \delta$~~

$$\boxed{\langle P_1(t)\rangle = -m\beta A^2\omega^2}$$

This is the average rate of energy loss due to the resistive force and clearly

$$\langle P_1(t)\rangle = \langle P\rangle$$

This makes sense - once the oscillator reaches a steady state and the transient dies out, the energy pumped in by the driving force is removed by the resistive force while keeping the mechanical energy of the system constant.

(c) $\langle P \rangle = m A^2 \omega^2 \beta$

$$= \frac{1}{m} \frac{\beta F_0^2 \omega^2}{4\beta^2\omega^2 + (\omega_0^2 - \omega^2)^2}$$

As ω is varied, $\langle P \rangle$ will be maximum when $\frac{d\langle P \rangle}{d\omega} = 0$ or equivalently when $\frac{d\langle P \rangle}{d\omega^2} = 0$

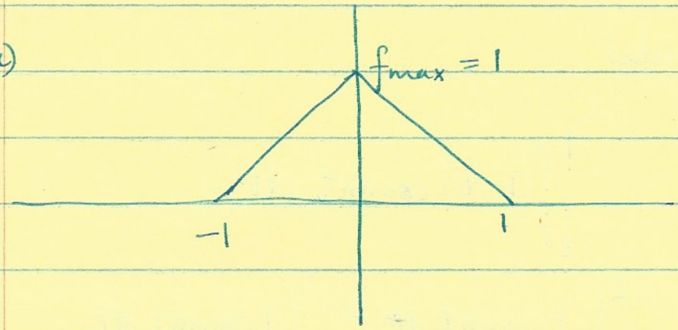
$$\frac{d\langle P \rangle}{d\omega^2} = \frac{\beta F_0^2}{m}\left[\frac{4\beta^2\omega^2 + (\omega_0^2-\omega^2)^2 - \omega^2\left[4\beta^2 - 2(\omega_0^2 - \omega^2)\right]}{\left\{4\beta^2\omega^2 + (\omega_0^2-\omega^2)^2\right\}^2}\right]$$

$$= \frac{\beta F_0^2}{m}\left[\frac{(\omega_0^2 - \omega^2)\left[(\omega_0^2+\omega^2)\cdot\right.}{\left\{4\beta^2\omega^2 + (\omega_0^2-\omega^2)^2\right\}^2}\right] = 0$$

$$\Rightarrow \quad \omega^2 = \omega_0^2 \Rightarrow \boxed{\omega = \omega_0}$$

So, $\langle P \rangle$ maximizes at $\omega = \omega_0$

5.49. (a)



Note that the period of $f(t) = 1-(-1) = 2 = \tau$. If the function is defined in the interval $(-1,1)$, it is defined for all t, and in the interval $(-1,1)$

$$f(t) = \begin{cases} t+1 & -1 < t < 0 \\ -t+1 & 0 \le t < 1 \end{cases}$$

Say the fourier series is given by

$$f(t) = \sum_{n=0}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t)$$

where $\omega = \frac{2\pi}{\tau} = \pi$

Then

$$a_n = \frac{\cancel{2}}{\cancel{2}} \int_{-1}^{1} f(t) \cos n\omega t \; dt$$

$$b_n = \int_{-1}^{1} f(t) \sin n\omega t \; dt$$

$$a_0 = \frac{1}{2} \int_{-1}^{1} f(t) \, dt$$

Plugging $f(t)$ into the formulae

$$a_n = \int_{-1}^{0} (1+t)\cos n\pi t \, dt + \int_{0}^{1} (1-t)\cos n\pi t \, dt$$

$$= \int_{-1}^{1} \cos n\pi t \, dt + \int_{-1}^{0} t\cos n\pi t \, dt - \int_{0}^{1} t\cos n\pi t \, dt$$

$$= 0 + \int_{-1}^{0} t\cos n\pi t \, dt - \int_{0}^{1} t\cos n\pi t \, dt$$

Substituting $t=-u$ in the $\int_{-1}^{0}$ integral,

$$a_n = \int_{1}^{0} (+u)\cos n\pi u \, du - \int_{0}^{1} t\cos n\pi t \, dt$$

$$= -\int_{0}^{1} u\cos(n\pi u)\, du - \int_{0}^{1} t\cos n\pi t \, dt$$

$$= -2\int_{0}^{1} u\cos(n\pi u)\, du$$

Using integration by parts

$$a_n = -2\left[\frac{u\sin \pi n u}{\pi n}\Big|_{0}^{1} - \int_{0}^{1} \frac{\sin(n\pi u)}{n\pi}\, du \right]$$

$$= \frac{2}{n\pi}\int_{0}^{1} \sin n\pi u \, du$$

$$= \frac{-2}{n\pi} \frac{\cos n\pi u}{n\pi}\Big|_{0}^{1}$$

$$= \frac{-2}{n^2\pi^2}\left[\cos n\pi - \cos 0\right]$$

$$\boxed{a_n = \frac{2}{n^2\pi^2}\left[1-(-1)^n\right]}$$

Now

$$b_n = \int_{-1}^{0} (1+t) \sin n\pi t\, dt + \int_{0}^{1} (1-t) \sin n\pi t\, dt$$

Substitute ~~Plug~~ in $t = -u$ in the first integral

$$b_n = \int_{1}^{0} (1-u) \sin n\pi u\, du + \int_{0}^{1} (1-t) \sin n\pi t\, dt$$

$$\Rightarrow \boxed{b_n = 0}$$

$$a_0 = \frac{1}{2} \int_{-1}^{1} f(t)\, dt$$

$$= \frac{1}{2} \left(\frac{1}{2}\right)(2)(1) = \frac{1}{2}$$

So,

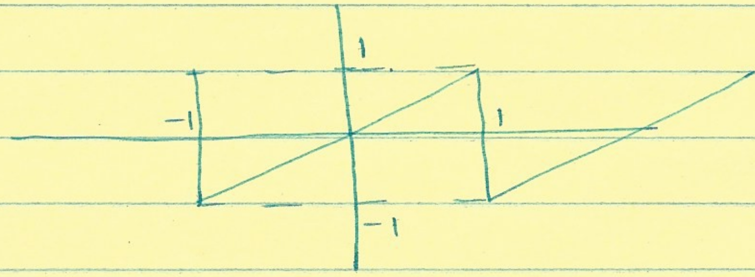
$$\boxed{\begin{aligned} a_0 &= 1/2 \\ a_n &= \left(\frac{2}{n\pi}\right)^2 \left[\frac{1-(-1)^n}{2}\right] \\ b_n &= 0 \end{aligned}}$$

So,

$$\cancel{f(t) = \frac{1}{2} + \sum_{n=0}^{\infty} \left(\frac{2}{n\pi}\right)^2}$$

$$f(t) = \frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{2}{n\pi}\right)^2 \left[\frac{1-(-1)^n}{2}\right] \cos(n\pi t)$$

(b)



Following the same procedure as in part (a),

$\tau = 2, \quad \omega = \pi$

$$f(t) = t \qquad -1 \le t \le 1$$

Say the fourier series is

$$f(t) = \sum_{n=0}^{\infty} \left(a_n \cos n\pi t + b_n \sin n\pi t\right)$$

$$a_n = \int_{-1}^{1} f(t) \cos n\pi t \, dt$$

$$= \int_{-1}^{1} t \cos n\pi t \, dt = 0 \quad \forall n.$$

$$b_n = \int_{-1}^{1} t \sin n\pi t \, dt$$

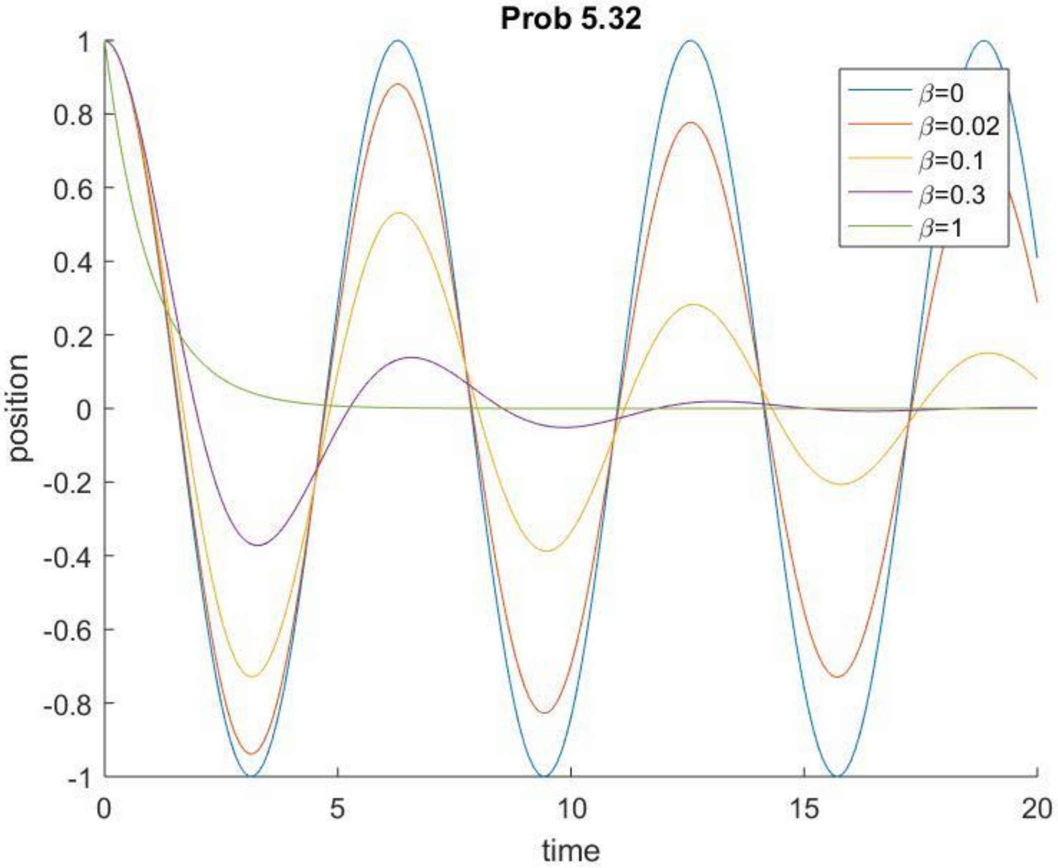
$$= -\frac{t \cos n\pi t}{n\pi}\Bigg|_{-1}^{1} + \int_{-1}^{1} \frac{\cos n\pi t}{n\pi} \, dt$$

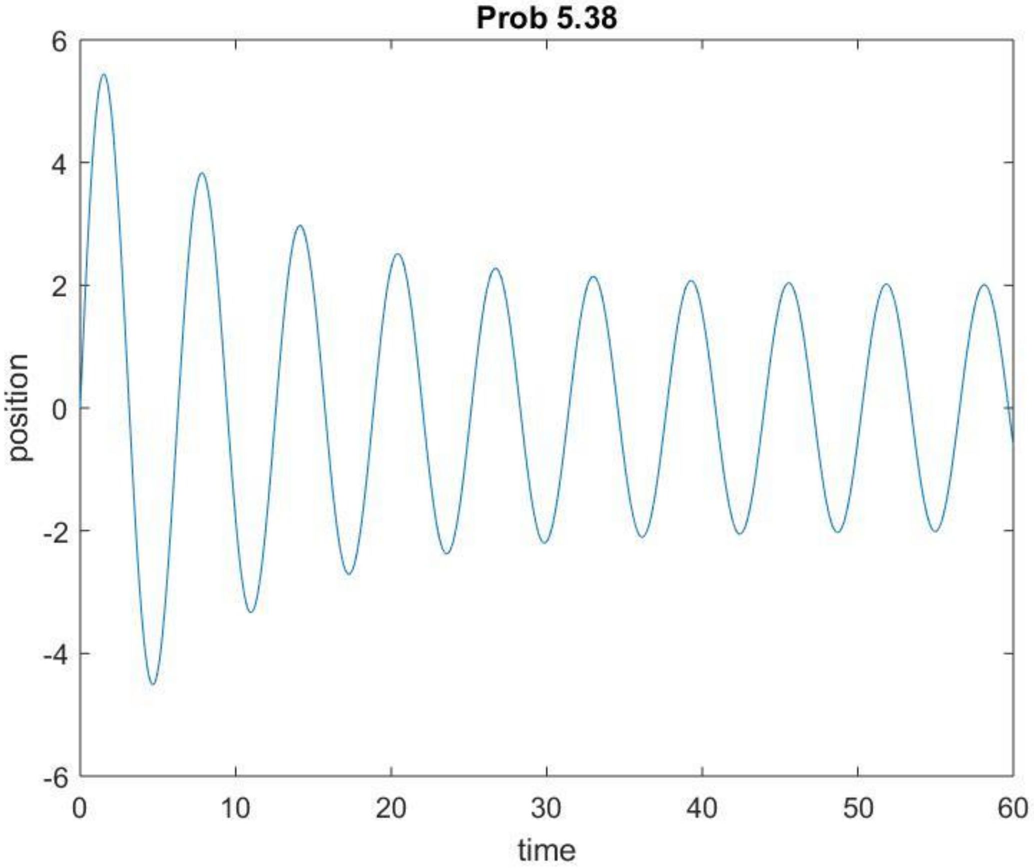
$$= \frac{-1}{n\pi}\left[2\cos n\pi\right] + \frac{1}{(n\pi)^2} \sin n\pi t \Bigg|_{-1}^{1}$$

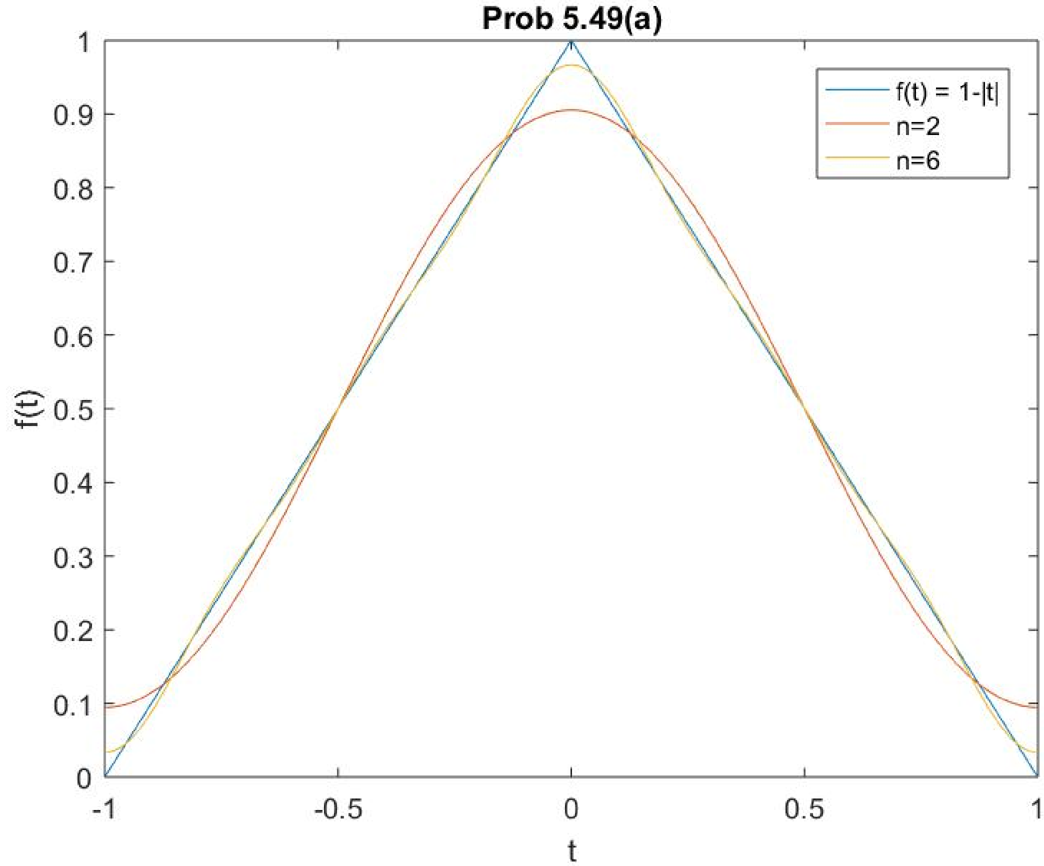
$$= \frac{-2\cos n\pi}{n\pi} = \frac{-2(-1)^n}{n\pi}$$

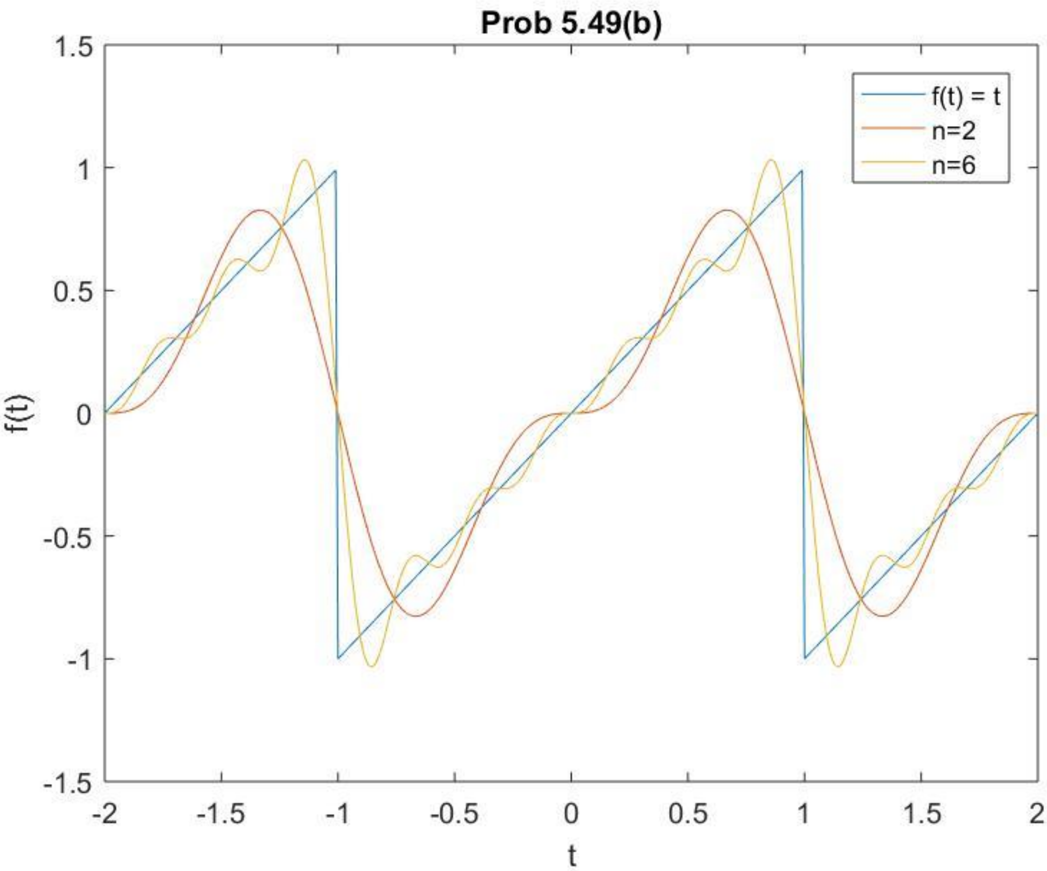
$$a_0 = \frac{1}{2}\int_{-1}^{1} f(t) \, dt = 0$$

$$\Rightarrow \quad \boxed{f(t) = -\sum_{n=1}^{\infty} \frac{2(-1)^n}{n\pi} \sin(n\pi t)}$$


Prob 5.32
position
time
β=0
β=0.02
β=0.1
β=0.3
β=1
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
5
10
15
20


Prob 5.38
position
time
6
4
2
0
-2
-4
-6
0
10
20
30
40
50
60


Prob 5.49(a)
f(t) = 1-|t|
n=2
n=6
f(t)
t
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-1
-0.5
0
0.5
1


Prob 5.49(b)
f(t) = t
n=2
n=6
f(t)
t
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2