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PHYS110A HW #7.

Taylor 6.1

A small displacement $d\theta$ in the $\hat{\theta}$ direction and $d\phi$ in $\hat{\phi}$ direction, the net displacement vector would be

$$d\vec{l} = R d\theta\, \hat{\theta} + R\sin\theta\, d\phi\, \hat{\phi}$$

Path lengths between two points (θ_1, ϕ_1) & (θ_2, ϕ_2) is then given by

$$L = \int_{(\theta_1,\phi_1)}^{(\theta_2,\phi_2)} |d\vec{l}|$$

$$|d\vec{l}| = R\sqrt{(d\theta)^2 + \sin^2\theta\, (d\phi)^2} = R\, d\theta \sqrt{1 + \sin^2\theta\, (d\phi/d\theta)^2}$$

$$\Rightarrow \quad L = R\int_{\theta_1}^{\theta_2} \sqrt{1+\sin^2\theta\, \phi'(\theta)^2}\, d\theta \quad , \quad \phi'(\theta) = \frac{d\phi}{d\theta}$$

Taylor 6.16

$$L = R\int_{\theta_1}^{\theta_2} \sqrt{1+\sin^2\theta\, \phi'(\theta)^2}\, d\theta$$

The integrand $f(\phi, \phi', \theta) = \sqrt{1+\sin^2\theta\, \phi'(\theta)^2}$

Using the Euler-Lagrange equation to solve for $\phi(\theta)$,

$$\frac{d}{d\theta}\left(\frac{\partial f}{\partial \phi'}\right) = \frac{\partial f}{\partial \phi}$$

Clearly $\frac{\partial f}{\partial \phi} = 0$

$$\Rightarrow \quad \frac{\partial f}{\partial \phi'} = c \quad , \text{ where } c \text{ is a constant}$$

$$\frac{\partial f}{\partial \phi'} = \frac{\partial}{\partial \phi'}\left(\sqrt{1+\sin^2\theta\,\phi'^2}\right)$$

$$= \frac{2\sin^2\theta\,\phi'}{2\sqrt{1+\sin^2\theta\,\phi'^2}}$$

$$\frac{\partial f}{\partial \phi'} = \frac{\sin^2\theta\,\phi'}{\sqrt{1+\sin^2\theta\,\phi'^2}} = c$$

Without loss of generality we can assume $(\theta_1, \phi_1) = (0, 0)$ (we can rotate the sphere so that one of the points is at the pole).

We plug in $\theta = 0$, ~~$\phi = 0$~~ into the equation to get

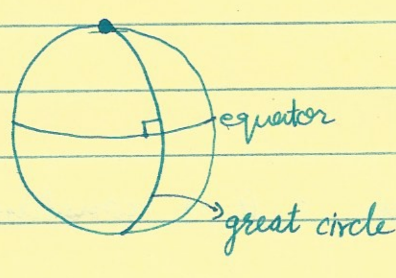
$$c = 0$$

$$\Rightarrow \quad \frac{\sin^2\theta\,\phi'}{\sqrt{1+\sin^2\theta\,\phi'^2}} = 0$$

$$\Rightarrow \quad \phi' = 0$$

$$\Rightarrow \quad \phi(\theta) = \phi_0 = \text{constant}.$$

So, the ~~sta~~ path of minimum distance is the great circle (or longitude)



Taylor 6.20

Say

$$f = f(y, y', x) = f(y, y') \text{ where } y \equiv y(x) \left(\begin{array}{c} y \text{ is a function} \\ \text{of } x \end{array} \right)$$

Then

$$\cancel{\frac{df}{dy}} \quad \frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(y(x+\Delta x), y'(x+\Delta x)) - f(y(x), y'(x))}{\Delta x}$$

$$\approx \lim_{\Delta x \to 0} \frac{f(y(x) + \Delta x\, y'(x), y'(x) + \Delta x\, y''(x)) - f(y(x), y'(x))}{\Delta x}$$

Using a first order taylor expansion,

$$f(y(x) + \Delta x\, y'(x), y'(x) + \Delta x\, y''(x))$$
$$\approx f(y(x), y'(x)) + \frac{\partial f}{\partial y} y'(x)\Delta x + \frac{\partial f}{\partial y'} y''(x)\Delta x$$

$$\Rightarrow \frac{df}{dx} \approx \lim_{\Delta x \to 0} \left[\frac{\partial f}{\partial y} y'(x) + \frac{\partial f}{\partial y'} y''(x) \right] \frac{\Delta x}{\Delta x}$$

$$\boxed{\frac{df}{dx} = y' \frac{\partial f}{\partial y} + y'' \frac{\partial f}{\partial y'}}$$

The Euler-Lagrange eqn is

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = 0$$

$$\Rightarrow \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = \frac{\partial f}{\partial y}$$

Plugging this into the boxed eqn,

$$\Rightarrow \quad \frac{df}{dx} = y'' \frac{\partial f}{\partial y'} + y' \left(\frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) \right)$$

$$= \frac{d}{dx} \left(y' \frac{\partial f}{\partial y'} \right)$$

$$\Rightarrow \quad \frac{d}{dx} \left(f - y' \frac{\partial f}{\partial y'} \right) = 0$$

$$\Rightarrow \quad f - y' \frac{\partial f}{\partial y'} = \text{constant}$$