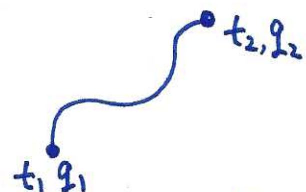
Lecture 14 The least action principle

The classical mechanics can be reformulated from a variational principle in analogy to the Fermat principle for geometric optics. Let the particle at t_1 and t_2 takes positions q_1 and q_2, then the particle moves in a way that the integral

$$\boxed{S = \int_{t_1}^{t_2} L(q, \dot{q}, t)\, dt}$$

takes the extremum. The S is called the action, and L only depends on coordinate and velocity.

Lagrangian



From the method of variation, the equation of motion can be solved as

$$\boxed{\frac{\partial L}{\partial q} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}}.$$

← Lagrangian equations

It can also be generalized to multiple coordinate systems with a set of coordinates and velocities (generalized coordinates and velocites).

$$L(q, \dot{q}, t) \longrightarrow L(q_i, \dot{q}_i, t) \qquad i = 1, 2, \cdots N$$

$$\boxed{\frac{\partial L}{\partial q_i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}}$$

Now we need to determine the form of L.

§ Determine Lagrangian — give the correct equations of motion

① Multiplying an arbitary ~~constant~~ constant to $\mathcal{L}$ does not change the equations of motion

Nevertheless, Lagrangian is additive — If two systems A and B are well separated such that interactions between them can be neglected, then

$$\mathcal{L} = \mathcal{L}_A + \mathcal{L}_B .$$

This property can reduce this arbitariness to an overall multiplication which is equivalent to a choice of unit of measurment.

② Consider two Lagrangians $L(q, \dot{q}, t)$ and $L'(q, \dot{q}, t)$. If their difference in a total time derivative of a function $f(q,t)$, i.e

$$\boxed{L'(q, \dot{q}, t) = L(q, \dot{q}, t) + \frac{d}{dt} f(q,t),}$$ then they

are equivalent. — give rise to the same equation of motion.

$$S' = \int_{t_1}^{t_2} L'(q, \dot{q}, t)\, dt = \int_{t_1}^{t_2} L(q \dot{q} t)\, dt + \int_{t_1} \frac{df}{dt}\, dt$$

$$\Rightarrow \quad S' = S + \underbrace{f(q_2, t_2) - f(q_1, t_1)}$$

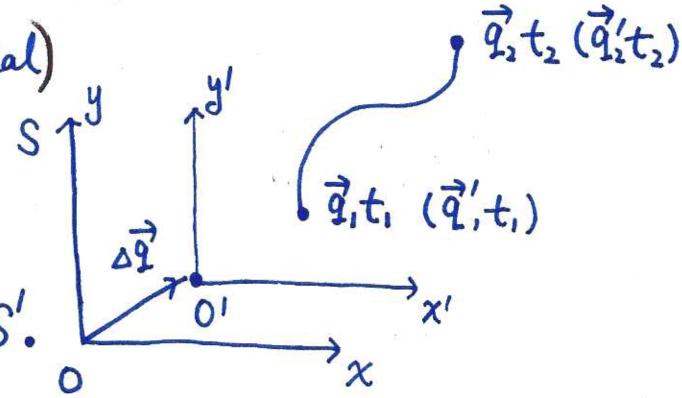
↓ fixed by boundary condition

$$\Rightarrow \delta S' = \delta S$$

§ Lagragian for a free particle – inertial frame

> A frame of reference can always be chosen in which space is homogeneous and isotropic, and time is homogeneous. — inertial fram

★ Frame translation —(spatial)



The event $(\vec q, t)$ in frame S

$\rightarrow$ $(\vec q', t') = (\vec q - \Delta\vec q, t)$ in frame S'.

The Lagrangians for the frame S and S', can be related

$$\boxed{L(\vec q, \dot{\vec q}, t) = L'(\vec q', \dot{\vec q}, t),}$$

(Since $\Delta\vec q$ is time independent, $\dot{\vec q}' = \dot{\vec q}$).

then we obtain the equivalent equations of motion for the process in S and S'.

If space is homogeneous, then we can not tell the difference of S and S',

i.e. an mechanical event can occur in S' frame as $(\vec q', \dot{\vec q}, t)$,

then it can also occur in S frame as $(\vec q', \dot{\vec q}, t)$.

it can be represented as

$$\boxed{L(\vec q', \dot{\vec q}, t) = L'(\vec q', \dot{\vec q}, t)}$$

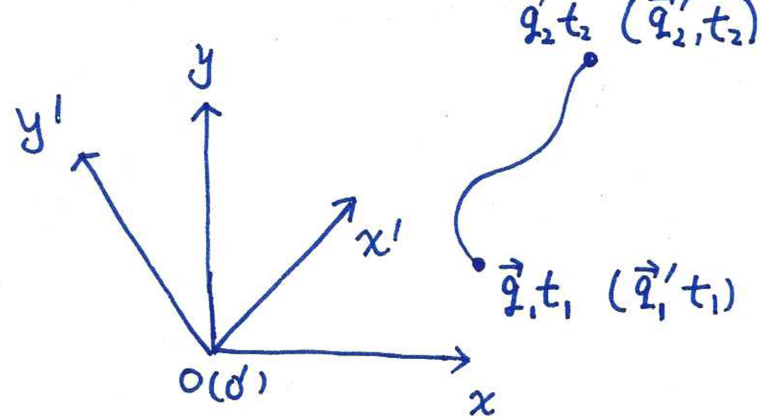
Since $L'(\vec{q}', \dot{\vec{q}}, t) = L(\vec{q}, \dot{\vec{q}}, t)$, we have

$$L(\vec{q}', \dot{\vec{q}}, t) = \boxed{L(\vec{q} - \Delta\vec{q}, \dot{\vec{q}}, t) = L(\vec{q}, \dot{\vec{q}}, t)}.$$

If $\Delta\vec{q}$ can take an arbitary value, then $\mathcal{L}$ does not depend on $\vec{q}$.

Similarly, time translation symmetry $\Rightarrow$ $\mathcal{L}$ does not depend on t.

Then for a free particle, its Lagrangian can only explicitly depend on $\dot{\vec{q}}$.



☆ Frame rotation

The event $(\vec{q}, t)$ in frame S

$\rightarrow$ $(\vec{q}', t') = (R^{-1}\vec{q}, t)$ in frame S', where R represents a rotation operation that rotates $S \rightarrow S'$. (R is typically represented by a 3×3 orthogonal matrix). Similarly, the velocity $\dot{\vec{q}}$ in frame S corresponds to $R^{-1}\dot{\vec{q}}$. Under this frame transformation, Lagrangian transforms as

$$\boxed{\mathcal{L}(\vec{q}, \dot{\vec{q}}, t) = \mathcal{L}'(\vec{q}', \dot{\vec{q}}', t) = \mathcal{L}'(R^{-1}\vec{q}, R^{-1}\dot{\vec{q}}, t)}$$

If space is isotropic, we also need

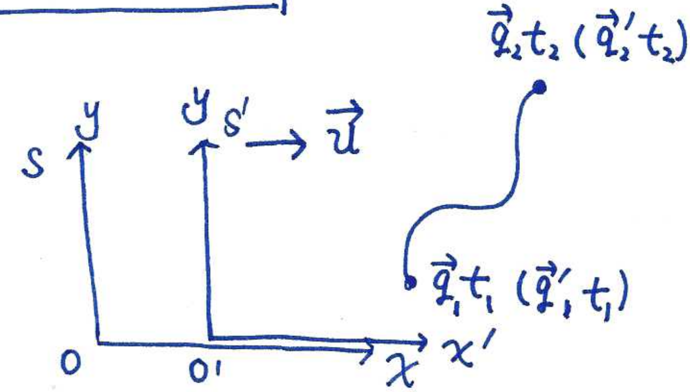
$$\boxed{\mathcal{L}(\vec{q}', \dot{\vec{q}}', t) = \mathcal{L}'(\vec{q}', \dot{\vec{q}}', t)}$$

$$\Rightarrow \mathcal{L}(\vec{q}, \dot{\vec{q}}, t) = \mathcal{L}(\vec{q}', \dot{\vec{q}}', t) = \mathcal{L}(\vec{q}', \dot{\vec{q}}', t)$$

or $$\boxed{\mathcal{L}(\vec{q}, \dot{\vec{q}}, t) = \mathcal{L}(R^{-1}\vec{q}, R^{-1}\dot{\vec{q}}, t)}$$

For free space $\mathcal{L} = \mathcal{L}[\dot{\vec{q}}]$, $\Rightarrow \boxed{\mathcal{L}(\dot{\vec{q}}) = \mathcal{L}[R^{-1}\dot{\vec{q}}]}$

If 'R' can take arbitary rotation, the $\mathcal{L}(\dot{\vec{q}})$ does not depend the orientation of $\dot{\vec{q}}$. $\Rightarrow \boxed{\mathcal{L}(\dot{\vec{q}}) = \mathcal{L}(\dot{q}^2)}$



☆ Galiean boost

The event $(\vec{q}\ t)$ in frame S

correspond to $(\vec{q}', t') = (\vec{q} - \vec{u} \cdot\ , t)$.

To yield the same equations of motion, actually we can release the condition between two different versions of Lagrangians as

$$\boxed{\mathcal{L}'(\vec{q}', \dot{\vec{q}}', t) = \mathcal{L}(\vec{q}, \dot{\vec{q}}, t) + \frac{df(\vec{q}, t)}{dt}}$$

This relation can ensure the equations of motion in S' frame in terms of $\vec{q}\,'$ and $\dot{\vec{q}}\,'$ is equivalent to that in frame S in terms of $\vec{q}$ and $\dot{\vec{q}}$.

The Galiean boost invariance means that a mechanical event with velocity $\dot{\vec{q}}\,'$ can occur in frame S', it can also occur in frame S. We write down $\boxed{\mathcal{L}'(\dot{q}'^2) = \mathcal{L}(\dot{q}'^2)}$

$$\Rightarrow \quad \mathcal{L}'(\dot{q}'^2) = \mathcal{L}(\dot{q}^2) + \frac{df(\vec{q},t)}{dt}$$

$$\mathcal{L}(\dot{q}'^2) = \boxed{\mathcal{L}(|\dot{\vec{q}} - \vec{u}|^2) = \mathcal{L}(\dot{q}^2) + \frac{df(\vec{q},t)}{dt}}$$

Now set $\vec{u}$ to be infinitesmall $\vec{\varepsilon}$, the

$$|\dot{\vec{q}} - \vec{\varepsilon}|^2 = \dot{q}^2 + \varepsilon^2 - 2\vec{\varepsilon}\cdot\frac{d}{dt}\vec{q}$$

$$\Rightarrow \mathcal{L}(|\dot{\vec{q}} - \vec{\varepsilon}|^2) - \mathcal{L}(\dot{q}^2) = -\frac{\partial \mathcal{L}}{\partial(\dot{q}^2)} \; 2\vec{\varepsilon}\cdot\frac{d}{dt}\vec{q}$$

If the right-hand side is a total derivative of time, then $\frac{\partial \mathcal{L}}{\partial(\dot{q}^2)}$ has to be a constant of time, we define

$$\boxed{\frac{\partial \mathcal{L}}{\partial(\dot{q}^2)} = \frac{1}{2}m},$$ then we arrive at for a free particle, it's Lagrangian

$\mathcal{L} = \frac{1}{2}m\dot{q}^2$, and if we have a system of non-interacting particles. we have

$$\boxed{\mathcal{L} = \sum_{\alpha} \frac{1}{2} m_\alpha \dot{q}_\alpha^2}.$$

§: Lagriangian of a single particle in an external field

We can add a spatial dependent part to the Lagrangian

$$\mathcal{L} = \frac{1}{2}m\dot{q}^2 - V_{ex}(\vec{q}, t),$$

then $$\delta S = 0 \Rightarrow m\ddot{\vec{q}} = \frac{\partial \mathcal{L}}{\partial \vec{q}} = -\nabla V_{ex}(\vec{q}, t)$$

Connecting to Newton's 2nd law, we can identify $V_{ex}(\vec{q}, t)$ as the external potential.

We can also generalize it to a system of interacting particle.

For simplicity, we neglect V_{ex}. Then

$$\mathcal{L} = \sum_\alpha \frac{1}{2} m_\alpha \dot{q}_\alpha^2 - \frac{1}{2}\sum_{\alpha\neq\beta} U(\vec{r}_i - \vec{r}_j) .$$

§ Change variables — generalized coordinates

$$\mathcal{L}(q_i, \dot{q}_i, t) \xrightarrow[Q_i = Q_i(q_1, \cdots q_n, t)]{} \tilde{\mathcal{L}}[Q_i, \dot{Q}_i, t] = \mathcal{L}[q_i(Q_j), \dot{q}_i(Q_j, \dot{Q}_j), t]$$

From the least action principle for $\tilde{\mathcal{L}}[Q_i, \dot{Q}_i, t]$, we should have

$$\boxed{\frac{\partial \tilde{\mathcal{L}}}{\partial Q_i} = \frac{d}{dt}\left(\frac{\partial \tilde{\mathcal{L}}}{\partial \dot{Q}_i}\right)} \leftarrow ②$$

Nevertheless, applying the least action principle for $\mathcal{L}(q_i, \dot{q}_i, t)$,

we have $\boxed{\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right)}$. ①

The question is that whether these two sets of equations are compatible with each other.

Let us start from ① and derive ②.

$$\frac{d}{dt}\left(\frac{\partial \tilde{\mathcal{L}}}{\partial \dot{Q}_i}\right) = \frac{d}{dt}\left[\sum_\alpha \frac{\partial \hat{\mathcal{L}}}{\partial \dot{q}_\alpha} \frac{\partial \dot{q}_\alpha}{\partial \dot{Q}_i}\right]$$

from $q_\alpha = q_\alpha(Q_1, \cdots Q_n, t) \Rightarrow \dot{q}_\alpha = \sum_\alpha \frac{\partial q_\alpha}{\partial Q_i} \dot{Q}_i + \frac{\partial q_\alpha}{\partial t}$

$$\Rightarrow \quad \frac{\partial \dot{q}_\alpha}{\partial \dot{Q}_i} = \frac{\partial q_\alpha}{\partial Q_i}$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial \tilde{\mathcal{L}}}{\partial \dot{Q}_i}\right) = \frac{d}{dt}\left(\sum_\alpha \frac{\partial \tilde{\mathcal{L}}}{\partial \dot{q}_\alpha}\frac{\partial q_\alpha}{\partial Q_i}\right) = \frac{d}{dt}\left(\sum_\alpha \frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha}\frac{\partial q_\alpha}{\partial Q_i}\right)$$

$$= \sum_\alpha \left[\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha}\right)\frac{\partial q_\alpha}{\partial Q_i} + \frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha}\frac{d}{dt}\left(\frac{\partial q_\alpha}{\partial Q_i}\right)\right]$$

$$= \sum_\alpha \left[\frac{\partial \mathcal{L}}{\partial q_\alpha}\frac{\partial q_\alpha}{\partial Q_i} + \frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha}\underbrace{\left(\sum_j \frac{\partial^2 q_\alpha}{\partial Q_i \partial Q_j}\dot{Q}_j + \frac{\partial^2 q_\alpha}{\partial Q_i \partial t}\right)}\right]$$

$$\frac{\partial}{\partial Q_i}\left(\sum_j \frac{\partial q_\alpha}{\partial Q_j}\dot{Q}_j + \frac{\partial q_\alpha}{\partial t}\right) = \frac{\partial}{\partial Q_i}\frac{d}{dt}q_\alpha$$

$$= \sum_\alpha \left(\frac{\partial \mathcal{L}}{\partial q_\alpha}\frac{\partial q_\alpha}{\partial Q_i} + \frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha}\frac{\partial \dot{q}_\alpha}{\partial Q_i}\right)$$

$$= \frac{\partial \mathcal{L}}{\partial Q_i} = \frac{\partial \tilde{\mathcal{L}}}{\partial Q_i} \quad \Rightarrow \quad \frac{d}{dt}\frac{\partial \tilde{\mathcal{L}}}{\partial \dot{Q}_i} = \frac{\partial \tilde{\mathcal{L}}}{\partial Q_i}$$

Thus the least action principle can be applied for different coordinates. The equations of motion are equivalent to each other.

Example: 2D motion in polar coordinates.

$$\mathcal{L} = T - U \quad : \quad T = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2)$$

$$U = U(r, \phi)$$

① $$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) \Rightarrow \begin{cases} m r \dot{\phi}^2 - \frac{\partial U}{\partial r} = \frac{d}{dt}(m \dot{r}) = m \ddot{r} \\ F_r = -\frac{\partial U}{\partial r} \end{cases}$$

$$\Rightarrow \boxed{F_r = m(\ddot{r} - r\dot{\phi}^2)}$$

② $$\frac{\partial \mathcal{L}}{\partial \phi} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) \Rightarrow -\frac{\partial U}{\partial \phi} = \frac{d}{dt}(m r^2 \dot{\phi}) = \frac{d L_z}{dt}$$

$$\nabla U = \frac{\partial U}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial U}{\partial \phi} \hat{\phi} \quad \text{(c.f. Prob 7.5)}$$

$$\vec{F} = -\nabla U \Rightarrow F_r = -\frac{\partial U}{\partial r}, \quad F_\phi = -\frac{1}{r}\frac{\partial U}{\partial \phi} \Rightarrow -\frac{\partial U}{\partial \phi} = r F_\phi = \Gamma$$

$$\Rightarrow \boxed{\Gamma = \frac{d L_z}{dt}}$$