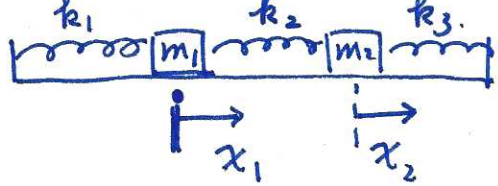
Lect 18: Oscillation (III), coupled ~~mode~~ — normal modes



§1: Two oscillators coupled by three springs. Assume m_1 and m_2 have displacements x_1 and x_2, respectively

$$F_1 = m_1 \ddot{x}_1 = -k_1 x_1 - k_2(x_1 - x_2) = -(k_1+k_2)x_1 + k_2 x_2$$

$$F_2 = m_2 \ddot{x}_2 = -k_3 x_2 - k_2(x_2 - x_1) = k_2 x_1 - (k_2+k_3) x_2$$

$$\Rightarrow \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = - \begin{pmatrix} \frac{k_1+k_2}{m_1} & \frac{-k_2}{m_1} \\ \frac{-k_2}{m_2} & \frac{k_2+k_3}{m_2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

consider a special case $k_1 = k_2 = k_3 = k$ and $m_1 = m_2 = m$

$$\Rightarrow \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -k/m \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

try $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{-i\omega_0 t} \Rightarrow \omega_0^2 \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = k/m \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

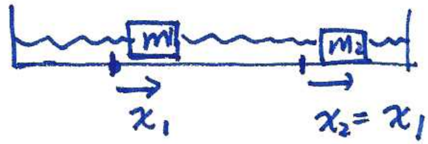
define $\omega_0^2 = k/m \, \lambda \Rightarrow \det \begin{bmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{bmatrix} = 0$

$$(\lambda-2)^2 - 1 = \lambda^2 - 4\lambda + 3 = 0 \Rightarrow \lambda_1 = 1, \quad \lambda_2 = 3$$

For 1st mode, $\lambda_1 = 1 \Rightarrow \begin{bmatrix} 2-1 & -1 \\ -1 & 2-1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0$

$\Rightarrow a_1 = a_2$ or $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-i\omega t}$.

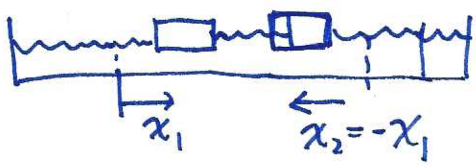
$\omega_1 = \sqrt{k/m}$



2nd mode $\lambda_2 = 3 \Rightarrow \begin{bmatrix} 2-3 & -1 \\ -1 & 2-3 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0$

$a_1 + a_2 = 0$ or $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-i\omega_2 t}$

$\omega_2 = \sqrt{3k/m}$



The general solution

$$X(t) = A_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-i\omega_1 t} + A_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-i\omega_2 t}$$

$\rightarrow$ take real part

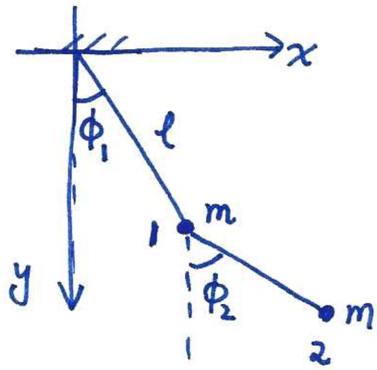
$$X(t) = |A_1| \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(\omega_1 t - \delta_1) + |A_2| \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(\omega_2 t - \delta_2)$$

* Normal coordinate $\xi_1 = \frac{1}{2}(x_1 + x_2)$, $\xi_2 = \frac{1}{2}(x_1 - x_2)$

$$\Rightarrow \begin{pmatrix} \ddot{\xi}_1 \\ \ddot{\xi}_2 \end{pmatrix} = -\frac{k}{m}\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \leftarrow$$

ξ_1, ξ_2 decoupled or diagonal.

§ Double pendulum



① $x_1 = l\sin\phi_1 \quad y_1 = l\cos\phi_1$

$$\Rightarrow v_1^2 = l^2[\cos^2\phi_1\,\dot\phi_1^2 + \sin^2\phi_1\,\dot\phi_1^2] = l^2\dot\phi_1^2$$

$$x_2 = l[\sin\phi_1 + \sin\phi_2] \quad y_2 = l[\cos\phi_1 + \cos\phi_2]$$

$$v_2^2 = [\dot x_2^2 + \dot y_2^2] = l^2\left[(\cos\phi_1\,\dot\phi_1 + \cos\phi_2\,\dot\phi_2)^2 + [\sin\phi_1\,\dot\phi_1 + \sin\phi_2\,\dot\phi_2]^2\right]$$

$$= l^2[\dot\phi_1^2 + \dot\phi_2^2 + 2\dot\phi_1\dot\phi_2\cos(\phi_1-\phi_2)]$$

$$\Rightarrow T = \frac{1}{2}ml^2\dot\phi_1^2 + \frac{1}{2}ml^2[\dot\phi_1^2 + \dot\phi_2^2 + 2\dot\phi_1\dot\phi_2\cos(\phi_1-\phi_2)]$$

$$= ml^2\dot\phi_1^2 + \frac{1}{2}ml^2\dot\phi_2^2 + ml^2\dot\phi_1\dot\phi_2\cos(\phi_1-\phi_2)$$

$$U = -mgl\cos\phi_1 - mgl[\cos\phi_1 + \cos\phi_2]$$

$$\Rightarrow \mathcal{L} = T - U = ml^2[\dot\phi_1^2 + \frac{1}{2}\dot\phi_2^2 + \dot\phi_1\dot\phi_2\cos(\phi_1-\phi_2)]$$

$$+ mgl[2\cos\phi_1 + \cos\phi_2]$$

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② keep to 2nd order $\dot\phi_1\dot\phi_2\cos(\phi_1-\phi_2) \simeq \dot\phi_1\dot\phi_2(1-\frac{1}{2}(\phi_1-\phi_2)^2) \simeq \dot\phi_1\dot\phi_2$

$$2\cos\phi_1 + \cos\phi_2 \simeq -\phi_1^2 - \frac{1}{2}\phi_2^2 + \text{const}$$

$$\Rightarrow \mathcal{L} = ml^2[\dot\phi_1^2 + \frac{1}{2}\dot\phi_2 + \dot\phi_1\dot\phi_2] - mgl\phi_1^2 - \frac{1}{2}mgl\phi_2^2$$

$$\frac{\partial L}{\partial \phi_1} = -2mgl\,\phi_1 \qquad \frac{\partial L}{\partial \dot\phi_1} = 2ml^2\dot\phi_1 + ml^2\dot\phi_2$$

$$\Rightarrow \frac{\partial L}{\partial \phi_1} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\phi_1}\right) \Rightarrow -2mgl\,\phi_1 = 2ml^2\ddot\phi_1 + ml^2\ddot\phi_2$$

or $$\boxed{2\ddot\phi_1 + \ddot\phi_2 = -2\frac{g}{l}\phi_1}$$

$$\frac{\partial L}{\partial \phi_2} = -mgl\,\phi_2 \qquad \frac{\partial L}{\partial \dot\phi_2} = ml^2\dot\phi_2 + ml^2\dot\phi_1$$

$$\frac{\partial L}{\partial \phi_2} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\phi_2}\right) \Rightarrow -mgl\,\phi_2 = ml^2\ddot\phi_2 + ml^2\ddot\phi_1$$

$$\Rightarrow \boxed{\ddot\phi_1 + \ddot\phi_2 = -g/l\,\phi_2}$$

$$\Rightarrow \frac{d^2}{dt^2}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} = -g/l\begin{pmatrix}2 & -1\\ -2 & 2\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}$$

3 Plugin $\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} = \begin{pmatrix}a_1\\ a_2\end{pmatrix}e^{-i\omega t} \Rightarrow$

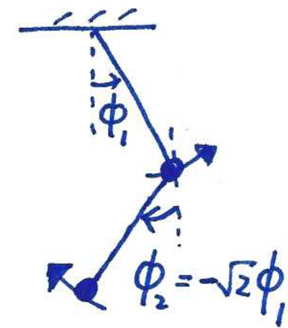
$$\omega^2\begin{pmatrix}a_1\\ a_2\end{pmatrix} = g/l\begin{pmatrix}2 & -1\\ -2 & 2\end{pmatrix}\begin{pmatrix}a_1\\ a_2\end{pmatrix} \Rightarrow g/l\begin{pmatrix}2-\lambda & -1\\ -2 & 2-\lambda\end{pmatrix}\begin{pmatrix}a_1\\ a_2\end{pmatrix} = 0$$

where $\omega^2 = g/l\,\lambda$

$$\Rightarrow \quad \det\begin{bmatrix} 2-\lambda & -1 \\ -2 & 2-\lambda \end{bmatrix} = 0 \quad \Rightarrow \quad (\lambda-2)^2 \mp 2 = 0$$

$$\lambda^2 - 4\lambda + 2 = 0 \Rightarrow \lambda = 2 \pm \sqrt{2}$$

$$\Rightarrow \quad \omega_1^2 = (2+\sqrt{2})\, g/L \qquad \omega_2^2 = (2-\sqrt{2})\, g/L$$



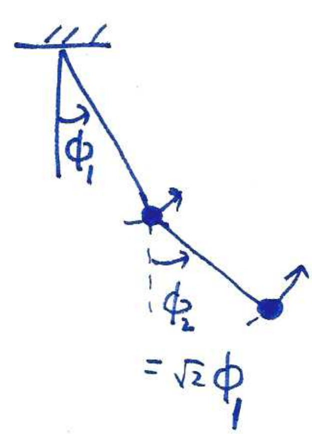
④ Normal modes

For $\omega_1^2 = (2+\sqrt{2})\, g/L \quad \Rightarrow \quad \begin{bmatrix} 2-(2+\sqrt{2}) & -1 \\ -2 & 2-(2+\sqrt{2}) \end{bmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0$

$$\begin{bmatrix} -\sqrt{2} & -1 \\ -2 & -\sqrt{2} \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0 \Rightarrow \sqrt{2}a_1 + a_2 = 0 \Rightarrow \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} e^{-i\omega t}$$

For $\omega_2^2 = (2-\sqrt{2})\, g/L \quad \Rightarrow \quad \begin{bmatrix} 2-(2-\sqrt{2}) & -1 \\ -2 & 2-(2-\sqrt{2}) \end{bmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \sqrt{2} & -1 \\ -2 & \sqrt{2} \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0$

$$\Rightarrow \quad a_2 = \sqrt{2}a_1 \qquad \Rightarrow \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} e^{-i\omega t}$$



§ The general case — quadratic Lagragian (harmonic approx)

$$T = \frac{1}{2}\sum_{jk} M_{jk}\, \dot{q}_j \dot{q}_k = \qquad q = (q_1 \cdots q_n)$$

$$\mathcal{U}(q) = \mathcal{U}(0) + \frac{1}{2}\sum_{jk} K_{jk}\, q_j q_k \quad \leftarrow \quad q=0 \text{ is the equilibrium position}$$

$$\mathcal{L} = T - \mathcal{U}(q)$$

$$\Rightarrow \quad \frac{\partial \mathcal{L}}{\partial q_i} = \sum_j K_{ij}\, q_j \qquad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = \frac{\partial \mathcal{L}}{\partial q_i}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \sum_j M_{ij}\, \dot{q}_j \qquad \Rightarrow \sum_j K_{ij}\, q_j = \sum M_{ij}\, \ddot{q}_j$$

$$\Rightarrow \quad M\ddot{q} = -Kq$$ where M and K are $n\times n$ matrices, and they are symmetr

set $q = \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} e^{-i\omega t} \Rightarrow \left[K - \omega^2 M\right]\begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} = 0$

or $\left[M^{-1}K - \omega^2\right]\begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} = 0$, then ω^2 is an eigenvalue of the matrix $M^{-1}K$,

and $\begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}$ is an eigen-vector $\leftarrow$ the normal modes!

For each eigen-frequency ω_i^2, we have the eigenvector $\begin{bmatrix} q_1^{(i)} \\ \vdots \\ q_n^{(i)} \end{bmatrix}$

Then the general solution is

$$q(t) = \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} = \cancel{\sum_i a_i \, e^{-i\omega_i t}}$$

$$= \sum_i A^{(i)} \begin{bmatrix} q_1^{(i)} \\ \vdots \\ q_n^{(i)} \end{bmatrix} e^{-i\omega_i t} \longrightarrow \text{finally take real part.}$$