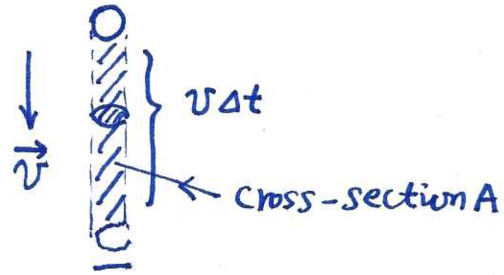
Lecture 3: Projectile with linear resistance

§: Air resistance

$\vec{f} = -f(v)\,\hat{v}$: always in the opposite direction of veolicity causing dissipation. (The dissipation power $\vec{f}\cdot\vec{v} < 0$)

How does the magnitude $f(v)$ behave? Let's analyze the origin of the drag force:



① The quadratic drag force.

In a short time Δt, the projectile travels at the distance $v\Delta t$. It pushes the air of the volume $vA\Delta t$ to the velocity of v. Thus

$$f_q \cdot \Delta t = \kappa \rho_a v A \Delta t \cdot v \quad \text{where } 0<\kappa<1 \text{ is a coefficient}$$

$$\Rightarrow \quad f_q = \kappa \rho_a v^2 A = \frac{\pi\kappa}{4}\rho_a v^2 D^2, \quad \text{where } D \text{ is the diameter.}$$

② Another linear drag due to Stokes' law of viscosity.

$$f_{lin} = 3\pi\eta D v, \quad \text{where } \eta \text{ is the viscosity of the fluid}$$

We can add ~~together~~ $f(v) = \beta D v + \gamma D^2 v^2$, where $\beta = 3\pi\eta$ and $\gamma = \frac{\kappa\pi}{4}\rho_a$. plug in $\eta = 1.7\times10^{-5}\ N\cdot Sm^{-2}$, $\rho_a = 1.29\ kg/m^3$, $\kappa = 1/4$ for spherical droplets in air, $\Rightarrow$ $\beta = 1.6\times10^{-4}\ N\cdot S/m^2$, $\gamma = 0.25\ N\,S^2/m^4$

$$\frac{f_q}{f_{lin}} = \frac{\gamma}{\beta} D v = \frac{\kappa}{12} \frac{\rho_a D v}{\eta}$$

plug in the values for air, $\Rightarrow$ $\frac{f_q}{f_{lin}} = \left[1.6\times10^3 \frac{s}{m^2}\right] D v$

$$\boxed{\frac{D v \rho_a}{\eta} = R \quad \text{Reynolds number}}$$

- For a base ball with $D = 7$ cm and $v = 5$ m/s

$$f_q / f_{lin} = 1.6\times10^3 \times 0.07 \times 5 \approx 600$$

- For rain drop with $D = 1$ mm, and $v = 0.6$ m/s

$$f_q / f_{lin} = 1.6\times10^3 \times 10^{-4} \times 0.6 \approx 1$$

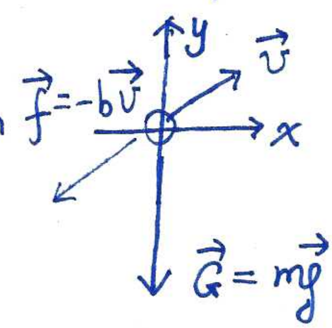
- For a milikan oil drop $D = 1.5\,\mu m$ and $v = 5\times10^{-5}$ m/s

$$f_q / f_{lin} = 1.6\times10^3 \times 1.5\times10^{-6} \times 5\times10^{-5} \approx 10^{-7}$$

$\Rightarrow$ For big and fast projectiles, the gudratic drag is more important. And for small and slow projectiles, the linear one is more important.

§: Motion with linear air resistance

$$m\ddot{\vec{r}} = m\vec{g} - b\vec{v} \quad \text{or} \quad m\dot{\vec{v}} = m\vec{g} - b\vec{v} \quad \text{where } b = \beta D.$$



$$\Rightarrow \begin{cases} m\dot{v}_x = -b v_x \\ m\dot{v}_y = mg - b v_y \end{cases}$$

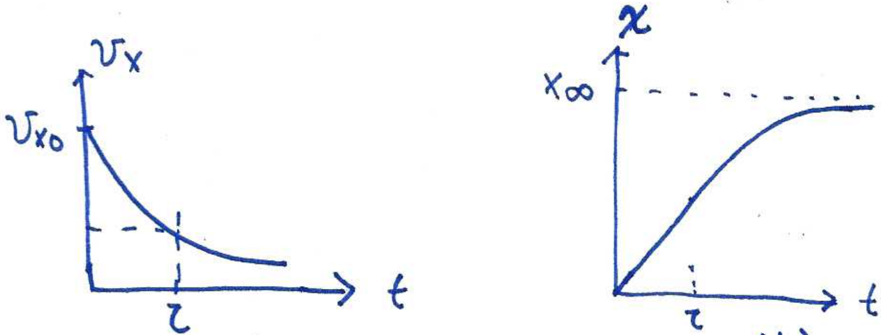
the motions in the x-direction and y-direction decouple.

In the x-direction, $v_x(t) = A e^{-t/\tau}$ where $\tau = m/b$.

A is determined by the initial velocity $A = v_{x,0}$.

$$x_\infty = v_{x0}\tau$$

$$x(t) = x(0) + \int_0^t v_{x,0}\, e^{-t'/\tau}\, dt' = x_\infty (1 - e^{-t/\tau}) \qquad (\text{set } x(0) = 0)$$



(a) (b)

"τ" is called the time const.

- Along the y-direction: it's an inhomogenous 1st order ODE.

$\dot{v}_y = g - b/m\, v_y$. It's solution $v_y(t) \xrightarrow{t\to\infty}$ const

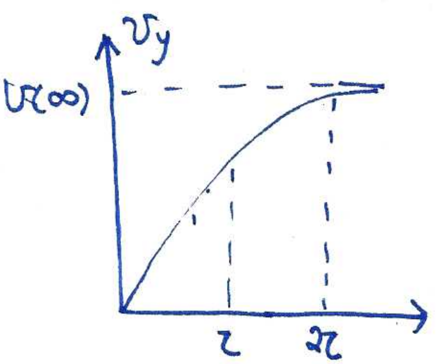
$$\Rightarrow \dot{v}_y(t\to+\infty) = 0 \Rightarrow v_y(t\to+\infty) = mg/b \triangleq v(\infty)$$

$$\Rightarrow \boxed{v_y(t) = v_{y,0}\, e^{-t/\tau} + v(\infty)(1 - e^{-t/\tau})}.$$

$$y(t) = \int_0^t v_y(t')\, dt' = v(\infty)\, t + (v_{y0} - v(\infty))\,\tau\,(1 - e^{-t/\tau})$$

(set $y(t=0) = 0$)

if ~~starting~~ with $v_y(t=0) = 0$, we have $v_y = v(\infty)(1 - e^{-t/\tau})$



time	percent of $v(\infty)$
τ	63%
2τ	86%
3τ	95%

Estimation of orders

1 $v(\infty) = \dfrac{mg}{b} = \dfrac{\rho\pi D^3 g}{6\beta D} = \dfrac{\rho\pi D^2 g}{6\beta}$

For an oil drop in Millikan experiment, $D = 1.5\,\mu m$. $\rho = 840\,kg/m^3$

$\Rightarrow v(\infty) = 6.1\times10^{-5}\,m/s$.

But for size $D = 0.2\,mm$, $\Rightarrow v(\infty) \simeq 1.3\,m/s$.
of drizzle drop

2 time scale : $\tau = m/b = \dfrac{v(\infty)}{g}$

For millikcan drop $\Rightarrow \tau \approx 6\times10^{-6}\,s$

For drizzle drop $\rightarrow \tau \approx 0.13\,s$.

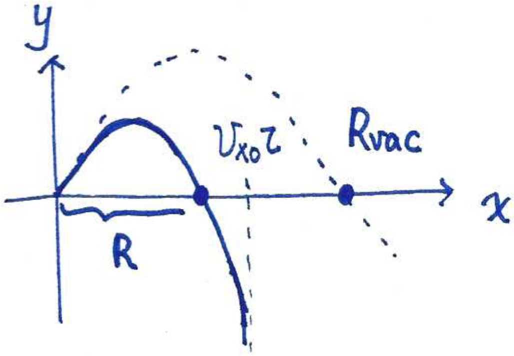
§ Trajectory and range

$$\left.\begin{aligned} &(1)\quad x(t) = v_{x0}\tau(1 - e^{-t/\tau}) \\ &(2)\quad y(t) = (v_{y0} + v(\infty))\tau(1 - e^{-t/\tau}) - v(\infty)t \end{aligned}\right\} \quad \text{with } x(0) = y(0) = 0.$$

Form (1) $\Rightarrow t = -\tau \ln\left(1 - \dfrac{x}{v_{x0}\tau}\right)$

$$\boxed{y = \frac{v_{y0} + v_{\infty}}{v_{x0}}\, x + v(\infty)\tau \ln\left(1 - \frac{x}{v_{x0}\tau}\right)}$$

x cannot exceed $x_{\infty} = v_{x0}\tau$.



The range on the horizontal direction in the vacuum

$$R_{vac} = \frac{2v_{x0}v_{y0}}{g}.$$

Now with resistance, we solve $\frac{v_{y0}+v(\infty)}{v_{x0}}R + v(\infty)\tau \ln\left(1-\frac{R}{v_{x0}\tau}\right) = 0$

In the limit of small resistance, τ is large, that $\frac{R}{v(\infty)\tau} \ll 1$

we use $\ln(1-\epsilon) = -\left(\epsilon + \frac{1}{2}\epsilon^2 + \frac{1}{3}\epsilon^3 + \cdots\right)$

$$\Rightarrow \frac{v_{y0}+v(\infty)}{v_{x0}}R - v(\infty)\tau\left[\frac{R}{v_{x0}\tau} + \frac{1}{2}\left(\frac{R}{v_{x0}\tau}\right)^2 + \frac{1}{3}\left(\underset{v_{x0}\tau}{R}\right)^3\right] \simeq 0$$

$$\Rightarrow \frac{v_{y0}}{v_{x0}}\frac{R}{v(\infty)\tau} = \frac{1}{2}\left(\frac{R}{v_{x0}\tau}\right)^2 + \frac{1}{3}\left(\frac{R}{v_{x0}\tau}\right)^3$$

$$\Rightarrow R = \frac{2v_{x0}v_{y0}}{g} - \frac{2}{3v_{x0}\tau}R^2$$

$$\Rightarrow \boxed{R \approx R_{vac} - \frac{2}{3v_{x0}\tau}\frac{4v_{x0}^2v_{y0}^2}{g^2} = R_{vac}\left[1 - \frac{4}{3}\frac{v_{y0}}{v(\infty)}\right]}$$

when $\frac{v_{y0}}{v(\infty)} \simeq 1$, the effect of air-resistance cannot be neglected!

Estimations: A metal pellets $D = 0.2\,mm$, $v = \cancel{0}\ 1\,m/s$ at angle 45°.

The range in the absence of resistance: $R_{vac} = \frac{2v_{x0}v_{y0}}{g} = \frac{v^2 \sin 2\theta}{g} = \frac{1}{9.8}\,m$

$\approx 10.2\,cm$

For gold $v(\infty) = \frac{\rho \pi D^2 g}{6\beta}$ plug in $\rho = 16\,g/cm^3$, $D = 0.2\,mm$

$\beta = 1.6 \times 10^{-4}\ N\cdot S/m^2$

$= 21\ m/s$

$\Rightarrow$ the correction $\frac{4}{3}\frac{1\times 0.7}{21} \approx 5\%$.

For Al, its density $\rho = 2.7\,g/cm^3$ which is about $1/6$ of gold.

thus the correction is about 6 times larger $\simeq 30\%$.