Lecture 8: More on energy

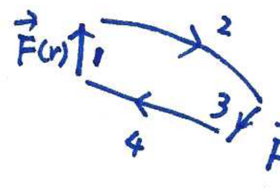
§ Central forces

- $\vec F(\vec r) = f(\vec r)\,\hat r$ — the forces point to the center.

If $\vec F(\vec r)$ is conservative, then it's spherically symmetric, and cnvesely, if $\vec F(\vec r)$ is spherically symmetric, then it's conservati

Proof: we have proved the 2nd half, and now we prove the 1st half.

If $\vec F(\vec r)$ is not spherically symmetric, then then there exist two directions along $\vec r$ and $\vec r\,'$ such at $f(\vec r) \neq f(\vec r\,')$. Construct the loop from $1 \to 2 \to 3 \to 4$. The paths 2 and 4 are perpendicular to $\vec F$,

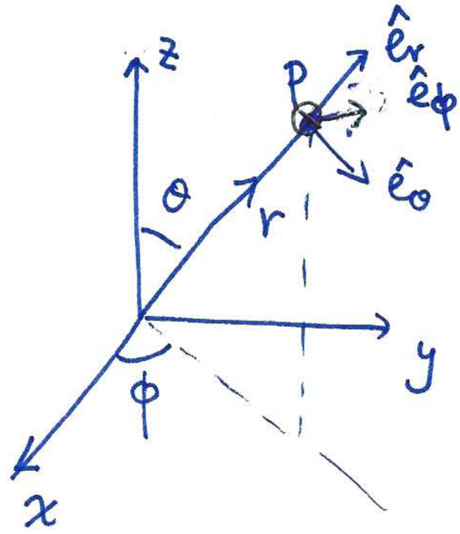


then $\oint \vec F \cdot d\vec l = \int_1 \vec F(\vec r)\cdot d\vec r + \int_3 d\vec r\,' \cdot \vec F(\vec r\,') = \int dr\,\left(f(r) - f(\vec r\,')\right) \neq 0$

Thus $\vec F$ cannot be conservative.

- spherical coordinate

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases}$$



$$\begin{cases} \hat{e}_r = \\ \hat{e}_\theta = \\ \hat{e}_\phi = \end{cases} \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix}$$

$$d\vec{r} = d(r\hat{e}_r) = dr\,\hat{e}_r + r\,d\hat{e}_r$$

$$d\hat{e}_r = d[\sin\theta\cos\phi]\,\hat{x} + d[\sin\theta\sin\phi]\,\hat{y} + d\cos\theta\,\hat{z}$$

$$= [\cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z}]\,d\theta$$

$$+[-\sin\theta\sin\phi\,\hat{x} + \sin\theta\cos\phi\,\hat{y}]\,d\phi = \hat{e}_\theta\,d\theta + \sin\theta\,d\phi\,\hat{\phi}$$

$$\Rightarrow d\vec{r} = dr\,\hat{e}_r + r\,d\theta\,\hat{e}_\theta + r\sin\theta\,d\phi\,\hat{e}_\phi$$

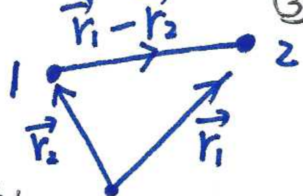
$$\Rightarrow df = \nabla f\cdot d\vec{r} = \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial \theta}d\theta + \frac{\partial f}{\partial \phi}d\phi$$

$$\Rightarrow \begin{cases} \nabla f\cdot\hat{e}_r = \dfrac{\partial f}{\partial r} \\ \nabla f\cdot r\,\hat{e}_\theta = \dfrac{\partial f}{\partial \theta} \\ \nabla f\cdot r\sin\theta\,\hat{e}_\phi = \dfrac{\partial f}{\partial \phi} \end{cases} \Rightarrow \boxed{\nabla f = \frac{\partial f}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{e}_\phi}$$

$\Rightarrow$ for centeral force field

$$\vec{F}(\vec{r}) = -\hat{e}_r\frac{\partial U}{\partial r}$$

§3 Energy of two interacting particles.



$$\begin{cases} \vec{F}_{12} = \vec{F}_{12}(\vec{r}_1-\vec{r}_2) \\ \vec{F}_{12} = -\vec{F}_{21} \end{cases}$$

— translation symmetry

interaction only depends on the relative displacement

If $\vec{F}_{12}$ with $\vec{r}_2$ fixed is

a conservative force on particle 1, ie. $\nabla_{\vec{r}_1} \times \vec{F}_{12} = 0$,

then we can express $\vec{F}_{12} = -\nabla_{\vec{r}_1} \mathcal{U}_{12}(\vec{r}_1-\vec{r}_2)$.

The same potential can also give rise to $\vec{F}_{21}$ through

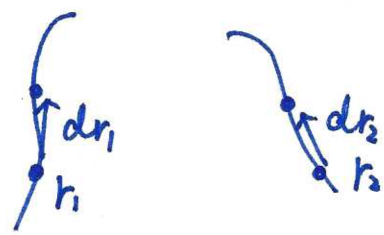
$$\boxed{\vec{F}_{21} = -\nabla_{\vec{r}_2} \mathcal{U}_{12}(\vec{r}_1-\vec{r}_2) = -\vec{F}_{12}}$$

← satisfying Newton's 3rd law.

Now let's apply the work-KE theorem to the two particle system:

$$dT_1 = d\vec{r}_1 \cdot \vec{F}_{12}$$

$$dT_2 = d\vec{r}_2 \cdot \vec{F}_{21}$$



define $T = T_1 + T_2 \Rightarrow dT = W_{tot}$

$$W_{tot} = d\vec{r}_1 \cdot \vec{F}_{12} + d\vec{r}_2 \cdot \vec{F}_{21} = (d\vec{r}_1 - d\vec{r}_2) \cdot \vec{F}_{12} = d(\vec{r}_1-\vec{r}_2) \cdot [-\nabla_1 \mathcal{U}_{12}(\vec{r}_1-\vec{r}_2)]$$

$$= -d\vec{r} \cdot \nabla \mathcal{U}_{12}(\vec{r}) = -d\mathcal{U}$$

where $\vec{r} = \vec{r}_1 - \vec{r}_2$ is the relative coordinate, and $\mathcal{U}$ is the interac

$$\Rightarrow \boxed{dE = 0, \text{ with } E = T_1 + T_2 + \mathcal{U}_{12}}$$

In principle, we can also inculde the external forces on 1 and 2, conservative

and introduce potentials $\mathcal{U}_1^{ex}$ and $\mathcal{U}_2^{ex}$, then

$$E = T_1 + T_2 + \mathcal{U}_1^{ex} + \mathcal{U}_2^{ex} + \mathcal{U}_{12}.$$

This process can be generalized to n-particle conservative systems, with

$$E = T_1 + T_2 + \cdots T_n + \mathcal{U}_1^{ex} + \mathcal{U}_2^{ex} + \cdots \mathcal{U}_n^{ex}$$
$$+ \mathcal{U}_{12} + \cdots \mathcal{U}_{1n} + \mathcal{U}_{23} + \cdots \mathcal{U}_{2n} + \cdots \mathcal{U}_{n-1,n}$$

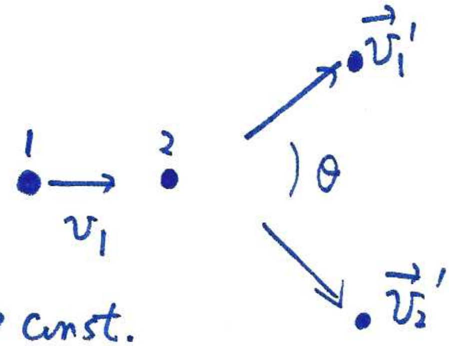
$$\Rightarrow \boxed{E = \sum_{i=1}^{n} \left(T_i + \mathcal{U}_i^{ex}\right) \leftarrow \text{single body} \quad + \sum_{i<j} \mathcal{U}_{ij} \leftarrow \text{interaction}}$$

double couting excluded!

Examples

① Equal mass, elastic collision

$m_1 = m_2$



The initial and finial configurations

$|\vec{r}_1 - \vec{r}_2| \to \infty$, thus $U(|\vec{r}_1 - \vec{r}_2|) \to$ const.

Energy conservation $\Rightarrow$ $T_{in} = T_f$. i.e $\frac{1}{2} m v_1^2 = \frac{1}{2} m v_1'^2 + \frac{1}{2} m v_2'^2$ (1

Momentum conservation $\quad m_1 \vec{v}_1 = m \vec{v}_1' + m \vec{v}_2'$ ②

From ① $\Rightarrow$ $v_1^2 = v_1'^2 + v_2'^2$

② $\Rightarrow$ $v_1^2 = v_1'^2 + v_2'^2 + 2\vec{v}_1' \cdot \vec{v}_2'$

$\Bigg\} \Rightarrow \vec{v}_1' \perp \vec{v}_2'$ or $\vec{v}_1' \cdot \vec{v}_2' = 0$.

② Rigid body: can be viewed as a cluster of particles

$$U^{int} = \sum_i \sum_{j>i} U_{ij}(|r_i - r_j|),$$ since $|r_i - r_j|$ fixed,

then $U^{int} \equiv$ fixed. $\Rightarrow$ We can apply energy conservatio

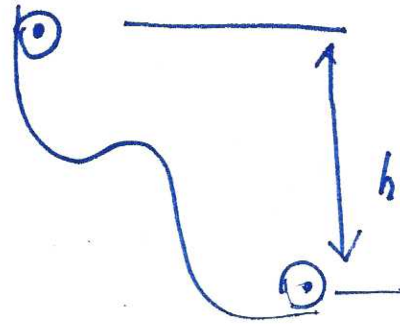
to rigid body as usual. But we need to inculde the rotation kinetic energy to the total kinetic energy.

The total kinetic energy of a cylinder is

$$\boxed{T = \frac{1}{2} M v^2 + \frac{1}{2} I \omega^2}$$

① $\frac{1}{2} M v^2$: the motion of the CM

② $\frac{1}{2} I \omega^2$: rotation aroun CM



$U^{ex} = Mg\,y$

initial: $v = 0$; $\omega = 0$; $y = h$

final: $\omega = v/R$, $y = 0$

$$\left.\begin{aligned} \Rightarrow \quad Mgh &= \tfrac{1}{2} M v^2 + \tfrac{1}{2} I \omega^2 \\ I &= \tfrac{1}{2} M R^2, \quad \omega = v/R \end{aligned}\right\} \Rightarrow Mgh = \tfrac{3}{4} M v^2$$

$$\Rightarrow v = \sqrt{\frac{4gh}{3}}$$